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A HANDBOOK

OF

PRACTICAL ASTRONOMY

FOR

UNIVERSITY STUDENTS AND ENGINEERS.

BY

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PREFACE

It has been the author's experience that the larger treatises on practical astronomy cannot be used satisfactorily with undergraduate students whose time is limited. He has presented the subject to large classes for several years from printed lecture notes. These are now written out in full and published in the following pages.

A chapter on the Surveyor's Transit has been added. This, with the closely related Chapters I, II and X, should enable the surveyor to determine the time, latitude and azimuth to an accuracy within the least reading of the circles of his instrument.

The illustrative examples, with very few exceptions, are based on observations made at one place, as would be the case with the student in an observatory, or with the observer in the field. The connections between the various problems with the same instrument are also more apparent than they would be if the observations were widely distributed.

An attempt has been made to give credit for methods used here which have not yet appeared in text-books.

Acknowledgement for valuable suggestions are due to Professors Holden and Schæberle of the Lick Observatory, and especially to Mr. W. J. Hussey, Instructor in Astronomy in the University of Michigan.

MT. HAMILTON CAL., 1891, October.

ERRATA.

Page 2,	line 40,	*For*	nenith	*read*	zenith.
" 26,	" 30,	"	semidiamater	*read*	semidiameter.
" 33,	" 20,	"	ascents	*read*	accents.
" 46,	" 4,	"	37	"	57.
" 55,	last line,	"	].	"	]*d*.
" 114,	line 17,	"	errror	"	error.
" 142,	last line,	footnote,		*read*	of § 139.

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PRACTICAL ASTRONOMY.

CHAPTER I.

THE CELESTIAL SPHERE.—SYSTEMS OF CO ORDINATES.

1. The heavenly bodies appear to us as if they were situated on the surface of a sphere of indefinitely great radius, whose center is at the point of observation. Their directions from us are constantly changing. They all appear to move from east to west at such a rate as to make one complete revolution in about twenty-four hours. This is due to the diurnal rotation of the earth. The sun moves eastward among the stars at such a rate as to make one revolution per year. This is caused by the annual revolution of the earth around the sun. The moon and the various planets have motions characteristic of the orbits which they describe. Measurements with instruments of precision enable us to detect other motions which, we shall see later, are conveniently divided into two classes: those due to parallax, refraction and diurnal aberration, which depend on the observer's geographical position; and those due to precession, nutation, annual aberration and proper motion, which are independent of the observer's position.

From data furnished by systematic observations it has been shown that these motions occur in accordance with well-defined physical laws. It is therefore possible to compute the very approximate positions of a celestial object for any given instants. A table giving at equal intervals of time the places of a body as affected by the second class of motions mentioned above, is called an *ephemeris* of the body. The astronomical annuals * furnish accurate ephemerides of the principal celestial objects several years in advance. Therefore, if an observer knows his

* The principal annuals are the *American Ephemeris and Nautical Almanac*, the *Berliner Astronomisches Jahrbuch*, the (British) *Nautical Almanac*, and the *Connaissance des Temps*. Unless otherwise specified we shall refer to the first of these, and call it the Nautical Almanac

position on the earth, he can, from data furnished by the Nautical Almanac, compute the direction of any star* at any instant. Conversely, by *observing* the directions of the stars with suitable instruments, he can determine the time and his geographical position. It is with this converse problem that we are concerned.

DEFINITIONS.

2. The sphere on whose surface the stars appear to be situated is called the *celestial sphere*. Any plane passing through the point of observation cuts the sphere in a great circle. Since the radius is indefinitely great, all parallel planes whose distances apart are finite cut the sphere in the same great circle.

In order to determine the position of a point on the sphere and express the relation existing between two or more points, the circles, lines, points and terms defined below are in current use.

The *horizon* is the great circle of the sphere whose plane is tangent to the earth's (theoretical) surface at the point of observation. It may also be defined as the great circle whose plane passes through the point of observation and perpendicular to the plumb-line.

The produced plumb-line, or *vertical line*, cuts the sphere above in the *zenith* and below in the *nadir*. The zenith and nadir are the poles of the horizon, and all great circles passing through them are called *vertical circles*.

The points of the horizon directly south, west, north and east of the observer are called, respectively, the *south*, *west*, *north* and *east points*.

The *meridian* is the vertical circle which passes through the south and north points.

The *prime vertical* is the vertical circle which passes through the east and west points.

The *altitude* of a point is its distance from the horizon, measured on the vertical circle passing through the point. Distances *above* the horizon are +, *below*, −. The altitudes of all points on the sphere are included between 0° and + 90°, and 0° and − 90°. Instead of the altitude, it is frequently convenient to use the *zenith distance*, which is the distance of the point from the zenith, measured on the vertical circle of the point. It is the complement of the altitude. The nenith distances of all points on the sphere lie between 0° and + 180°.

*For convenience we shall use *star* or *point* to denote any celestial object.

The *azimuth* of a point is the arc of the horizon intercepted between the vertical circle of the point and some fixed point assumed as origin. With astronomers it is customary to reckon azimuth from the south point around to the west through 360°. Surveyors frequently reckon from the north point.

The *celestial equator* is the great circle of the sphere whose plane is perpendicular to the earth's axis. It therefore coincides with or is parallel to the terrestrial equator.

The earth's axis produced is the *axis of the celestial sphere.* It cuts the sphere in the *north* and *south poles* of the equator. We shall for brevity call them the *north* and *south poles.*

All great circles passing through the north and south poles are called *hour circles,* or *circles of declination.* The hour circle passing through the zenith coincides with the meridian.

The *declination* of a point is its distance from the equator, measured on the hour circle passing through the point. Distances *north* are +; *south,* —. The declinations of all points on the sphere are included between 0° and + 90°, and 0° and — 90°.

Instead of the declination, it is sometimes convenient to use the *north polar distance*, which is the distance of a point from the north pole, measured on the hour circle of the point. It is therefore the complement of the declination. The north polar distances of all points lie between 0° and + 180°.

The *hour angle* of a point is the arc of the equator intercepted between the meridian, or south point of the equator, and the hour circle passing through the point. In practice, however, it is customary to consider the hour angle as the equivalent angle at the north pole between the meridian and hour circle. It is reckoned from the meridian around to the west through 24 hours, or 360°.

The *ecliptic* is the great circle of the sphere formed by the plane of the earth's orbit; or, it is the great circle described by the apparent annual motion of the sun. It intersects the equator in two points called the *equinoxes.*

The *vernal equinox* is that point through which the sun appears to pass in going from the south to the north side of the equator (about March 20).

The *autumnal equinox* is that point through which the sun appears to pass in going from the north to the south side of the equator (about Sept. 22).

The *solstices* are the points of the ecliptic 90° from the

equinoxes. The sun is in the *summer solstice* about June 21; in the *winter solstice* about Dec. 21.

The *equinoctial colure* is the hour circle passing through the equinoxes. The *solstitial colure* is the hour circle passing through the solstices.

The angle between the equator and ecliptic is called the *obliquity of the ecliptic.*

The *right ascension* of a point is the arc of the celestial equator intercepted between the vernal equinox and the hour circle of the point. It is measured from the vernal equinox toward the east through 24 hours, or 360°.

Great circles perpendicular to the ecliptic are called *latitude circles.*

The *latitude* of a point is its distance from the ecliptic, measured on the latitude circle passing through the point. Distances *north* are +; *south,* —. The latitudes of all points on the sphere are included between 0° and + 90°, and 0° and — 90°.

The *longitude* of a point is the arc of the ecliptic intercepted between the vernal equinox and the latitude circle of the point. It is measured from the vernal equinox toward the east through 360°.

The position of an observer on the earth's surface is defined by his geographical latitude and longitude.

The *geographical latitude* of a place is the declination of the zenith of the place. It is also equal to the altitude of the north pole. Latitudes of places *north* of the equator are +; *south,* —.

The *geographical longitude* of a place is the arc of the equator intercepted between the meridian of the place and the meridian of some other place assumed as origin. It is customary to reckon longitudes *west* (+) and *east* (—) from the meridian of Greenwich, through 12 hours, or 180°.

The preceding definitions are illustrated by Fig. 1. The celestial sphere is orthogonally projected on the plane of the horizon, $SWNE$. The zenith Z is projected on the point of observation; NZS is the meridian; EZW the prime vertical; $WVQE$ the equator; $VLBV'$ the ecliptic; P the north pole; P' the north pole of the ecliptic; V the vernal equinox; V' the autumnal equinox; VP the equinoctial colure; CPP' the solstitial colure; $BC = PP' = BVC =$ the obliquity of the ecliptic.

Let O be any point on the sphere; then ZOA is its vertical circle; MOP its hour circle; LOP' its latitude circle. The point O is defined by the following arcs, called *spherical co-ordinates:*

AO = Altitude, h;
ZO = Zenith distance, z;
$SA = SZA$ = Azimuth, A;
MO = Declination, δ;
PO = North Polar Distance, P;
$QM = QPM$ = Hour angle, t;
VM = Right ascension, α, or $R\ A$;
LO = Latitude, β;
VL = Longitude, λ.

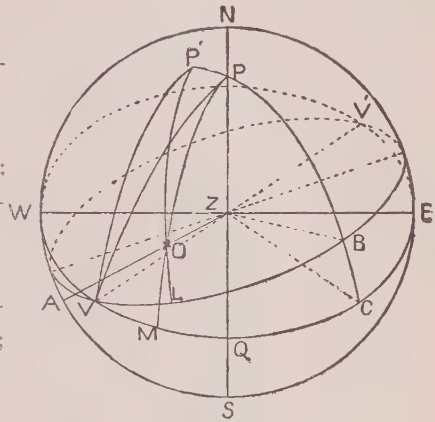

Fig. 1.

3. It will be observed that the horizon, equator and ecliptic are of fundamental importance. They are called *primitive circles.* Vertical circles, hour circles and latitude circles, which are respectively perpendicular to them, are called *secondaries.* Only two spherical co-ordinates, one measured on a primary circle, the other on its secondary, are necessary to completely determine the direction of a point; and from the definitions just given, to meet the requirements of astronomical work, we formulate four

SYSTEMS OF CO-ORDINATES.

System.	Circles of Reference.		Co-ordinates.	
	Primary.	Secondary.	Primary.	Secondary.
I.	Horizon.	Vertical circle.	Azimuth.	Altitude.
II.	Equator.	Hour circle.	Hour angle.	Declination.
III.	Equator.	Hour circle.	R't ascension.	Declination.
IV.	Ecliptic.	Latitude circle.	Longitude.	Latitude.

The altitude, azimuth and hour angle of a star are continually changing. They are functions of the time and the observer's position. Hence they are adapted to the determinations of time, azimuth, and geographical latitude and longitude. Right

ascension and declination are independent of the observer's position, and nearly independent of the time. They are largely used for recording the relative positions of stars, and in ephemerides. Latitude and longitude are also independent of the observer's position, but are employed almost exclusively in theoretical astronomy.

CHAPTER II.

TIME.

4. The passage of any point of the celestial sphere across the meridian of an observer is called the *transit*, or *culmination*, or *meridian passage* of that point. In one rotation of the sphere about its axis, every point of it is twice on the meridian; once at *upper* culmination (above the pole), and once at *lower* culmination (below the pole). For an observer in the northern hemisphere, a star whose north polar distance is less than the latitude is constantly above the horizon, and both culminations are visible ; a star whose south polar distance is less than the latitude is constantly below the horizon, and both culminations are invisible ; and a star between these limits is visible at upper culmination, but invisible at the lower. For an observer in the southern hemisphere the first two cases are reversed.

5. The time element in practical astronomy is very important. The astronomical day begins at noon and is reckoned from 0 hours to 24 hours. Thus, Feb. 1, 10^h A. M., civil reckoning, is Jan. 31^d 22^h astronomical time.

Three systems of time are in common use: *sidereal*, *apparent* (or *true*) *solar*, and *mean solar*.

A *sidereal day* is the interval of time between two successive transits of the true vernal equinox over the same meridian. The *sidereal time* at any instant is therefore the hour angle of the vernal equinox at that instant; or, it is the right ascension of the observer's meridian at that instant. It follows, then, that a star whose right ascension is α will be on the meridian at α hours sidereal time. The rotation of the earth on its axis is perfectly uniform; but owing to precession and nutation the vernal equinox has a minute and irregular motion to the west, amounting on the average to $0''.126$ per day: so that a sidereal day does not correspond exactly to one rotation of the earth, nor is its length absolutely uniform, but it is *sensibly* so.

An *apparent solar day* is the interval of time between two successive upper transits of the sun over the same meridian.

The hour angle of the sun at any instant is therefore the *apparent time* at that instant. But the apparent day varies greatly in length, for two reasons, viz.:

First,—The earth moves in an ellipse with a variable velocity. Hence the sun's (apparent) eastward motion (in longitude) is variable.

Second,—The sun's (apparent) motion is in the ecliptic. Hence its motion in right ascension and hour angle is variable, and a clock cannot be rated to keep apparent time.

A convenient solar time is obtained in this way: Assume an imaginary body to move in the ecliptic with a uniform angular velocity such that it and the sun pass through perigee at the same instant. Assume a second imaginary body to move in the equator with a uniform angular velocity such that the two will pass through the vernal equinox at the same instant. The second body is called the *mean sun*.

A *mean solar day* is the interval of time between two successive upper transits of the mean sun over the same meridian. The hour angle of the mean sun is the *mean time*.

The difference between the apparent and mean time is called the *equation of time*. Its value is given in the Nautical Almanac for the instants of Greenwich apparent and mean noon and Washington apparent noon, whence its value may be obtained for any other instant by interpolation.

6. The interval of time between two successive passages of the mean sun through the mean vernal equinox—called a tropical year—was for the year 1800, according to Bessel, 365.24222* mean solar days.

The number of sidereal days in this interval is 366.24222, since the vernal equinox makes one more transit over any given meridian than the sun. Therefore,

$$365.24222 \text{ mean days} = 366.24222 \text{ sidereal days.}$$

Whence

$$\begin{aligned} 24^h \text{ mean time} &= 24^h\ 3^m\ 56^s.555 \text{ sid. time,} \\ 24 \text{ sid. time} &= 23\ \ 56\ \ \ 4\ .091 \text{ mean time.} \end{aligned}$$

From these equations it is found that the gain of sidereal time on mean time in one mean hour is $9^s.8565$; and in one sidereal hour, $9^s.8296$. These are the amounts by which the right ascension of the mean sun increases in one mean and one sidereal hour, respectively.

* The length of the year is diminishing at the rate of $0^s.595$ per century.

CONVERSION OF TIME.

7. In nearly every problem of practical astronomy it is necessary to convert the time at one place into the corresponding time at another place, or to convert the time in one system into the corresponding time in another system. By means of the data furnished by the Nautical Almanac this is readily done.

8. *To convert the time at one place into the corresponding time at another.*

Since time is defined as an hour angle, the difference of time at two places is the difference of the two corresponding hour angles; which is the difference of their longitudes. Therefore, if the difference of longitude be added to the time at the western place the sum is the corresponding time at the eastern. If it be subtracted from the time at the eastern the result is the time at the western.

Example 1. The Ann Arbor mean time is 1891 March 10^d $21^h\ 10^m\ 54^s.7$. What is the corresponding Greenwich mean time?

Ann Arbor mean time,	1891, March	$10^d\ 21^h\ 10^m\ 54^s.7$
Longitude Ann Arbor, Naut. Alm., p. 482,		$+\ 5\ \ 34\ \ 55\ .14$
Greenwich mean time,	1891, March	$11\ \ 2\ \ 45\ \ 49\ .84$

Example 2. The Washington sidereal time is $0^h\ 23^m\ 17^s.10$. What is the corresponding Ann Arbor sidereal time?

Washington sidereal time,	$0^h\ 23^m\ 17^s.10$
Longitude Ann Arbor, Naut. Alm., p. 482,	$+\ 0\ \ 26\ \ 43\ .10$
Ann Arbor sidereal time,	$23\ \ 56\ \ 34\ .00$

9. *To convert apparent time at a given place into mean time, and vice versa.*

This requires the equation of time at the given instant, which applied with the proper sign to the one gives the other. If apparent time is given, convert it into Greenwich apparent time, and take the equation of time from page I of the given month in the Nautical Almanac. If mean time is given, convert it into Greenwich mean time, and take the equation of time from page II of the month.

In taking this and other data from the Almanac, care must be exercised in making the interpolations. Thus, let it be required to determine the equation of time at Greenwich apparent time 1891, Feb. $24^d\ 10^h$. Its value for apparent noon is $+\ 13^m\ 25^s.52$, and the difference for one hour *at* noon is $0^s.381$. The difference for one hour at noon the next day is $0^s.406$. The hourly difference is, therefore, variable, but we

may assume the second difference constant. The change in the equation during the 10 hours is ten times the *average* hourly change for the 10 hours; that is, the hourly change *at the middle period*, or five hours after noon. The average hourly change is $0^s.386$, and the desired equation of time is

$$+ 13^m\ 25^s.52 - 10 \times 0^s.386 = + 13^m\ 21^s.66.$$

Example. The Berlin mean time is 1891, Feb. $28^d\ 0^h\ 11^m\ 20^s.6$. What is the apparent time?

Berlin mean time,	1891 Feb. $28^d\ 0^h\ 11^m\ 20^s.6$
Longitude Berlin,	— 0 53 34 91
Greenwich mean time,	Feb. 27 23 17 45 .69

This is $23^h.296$ after Gr. mean noon Feb. 27, or $0^h.704$ before noon Feb. 28. In this and similar cases the interpolation should be made for the interval before noon.

Equation of time, Gr. mean noon, Feb. 28,	$-12^m\ 44^s.52$
Change before noon, $0.704 \times 0^s.473$	0 .33
Equation of time,	— 12 44 .85
Berlin apparent time,	1891 Feb. $27^d\ 23^h$ 58 35 .75

10. *To convert a mean time interval into the equivalent sidereal interval, and vice versa.*

In § 6 it is shown that sidereal time gains $9^s.8565$ on mean time in one mean hour. The corresponding gain for any number of hours, minutes and seconds, is tabulated in Table III of the appendix to the Nautical Almanac. If this gain be added to the mean time interval, the sum is the equivalent sidereal interval.

The gain of sidereal time on mean time in one sidereal hour is $9^s.8296$. The corresponding gain for any number of hours, minutes and seconds, is tabulated in Table II of the appendix to the Nautical Almanac. If this gain be subtracted from the sidereal interval the difference is the equivalent mean time interval.

Example 1. A mean time interval is $17^h\ 33^m\ 21.76^s$. Find the corresponding sidereal interval.

Mean time interval,	$17^h\ 33^m\ 21^s.76$
Gain of sid. on mean, Table III,	2 53 .04
Sidereal interval,	17 36 14 .80

Example 2. A sidereal time interval is $17^h\ 36^m\ 14^s.80$. Find the corresponding mean time interval.

Sidereal interval,	$17^h\ 36^m\ 14^s.80$
Gain of sid. on mean, Table II,	2 53 .04
Mean time interval,	17 33 21 .76

11. *To convert mean time into sidereal time.*

Mean time at any instant is the interval after mean noon. If this be converted into the equivalent sidereal interval and added to the sidereal time at noon, the sum is the sidereal time required. The sidereal time at noon is equal to the right ascension of the mean sun at that instant. The Nautical Almanac gives the mean sun's right ascension at Greenwich mean noon, whence its right ascension at noon for a place whose longitude is L may be obtained by adding the term $L \times 9^s.8565$.

Example. The Ann Arbor mean time is 1891 Feb. 20^d $11^h\ 45^m\ 20^s.4$. What is the equivalent sidereal time?

Right ascension mean sun, Gr. mean noon,	$22^h\ 0^m\ 31^s.75$
Change in $5^h\ 34^m\ 55^s.14$, Table III,	0 55 .02
Right ascension, or sid. time, Ann Arbor noon,	22 1 26 .77
Mean time interval,	11 45 20 .4
Gain of sid. on mean, Table III,	1 55 87
Equivalent sidereal interval,	11 47 16 .27
Sidereal time,	9 48 43 .04

12. *To convert sidereal time into mean time.*

If the sidereal time of the preceding mean noon (formed as before) be subtracted from the given time, the result is the sidereal interval after mean noon. This interval converted into the equivalent mean time interval is the mean time desired.

Example. On Feb. 20, 1891, the sidereal time at Ann Arbor is $9^h\ 48^m\ 43^s.04$. What is the mean time?

Ann Arbor sid. time, Feb. 20, 1891,		$9^h\ 48^m\ 43^s.04$
R. A. mean sun preceding Gr. mean noon,		22 0 31 .75
Reduction for longitude, Table III,		55 02
Sid. time, Ann Arbor mean noon,		22 1 26 .77
Sid. interval after mean noon,		11 47 16 .27
Gain of sid. on mean, Table II,		1 55 .87
Ann Arbor mean time,	1891 Feb. 20^d	11 45 20 .40

CHAPTER III.

TRANSFORMATION OF SPHERICAL CO-ORDINATES.

13. *Given the altitude and azimuth of a star, required its declination and hour angle.*

This transformation is effected by solving the spherical triangle PZO, Fig. 1, whose vertices are at the pole, the star,

and the zenith. Three parts of this triangle are known: ZO the zenith distance or complement of the given altitude, PZO the supplement of the given azimuth, and PZ the complement of the given latitude; from which, by the methods of Spherical Trigonometry, we can find PO the complement of the required declination, and ZPO the required hour angle.

For any spherical triangle ABC we have [*Chauvenet's Sph. Trig.* § 114] the general equations

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{1}$$
$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A, \tag{2}$$
$$\sin a \sin B = \sin b \sin A. \tag{3}$$

To adapt these equations to the triangle POZ, let

$$\begin{aligned} \mathrm{A} &= PZO = 180° - A, & \mathrm{a} &= PO = 90° - \delta, \\ \mathrm{b} &= ZO = 90° - h, & \mathrm{B} &= ZPO = t, \\ \mathrm{c} &= PZ = 90° - \varphi. \end{aligned}$$

and (1), (2) and (3) become

$$\sin \delta = \sin h \sin \phi - \cos h \cos \phi \cos A, \tag{4}$$
$$\cos \delta \cos t = \sin h \cos \phi + \cos h \sin \phi \cos A, \tag{5}$$
$$\cos \delta \sin t = \cos h \sin A, \tag{6}$$

which enable us to find δ and t.

If h be replaced by $90° - z$ these become

$$\sin \delta = \cos z \sin \phi - \sin z \cos \phi \cos A, \tag{7}$$
$$\cos \delta \cos t = \cos z \cos \phi + \sin z \sin \phi \cos A, \tag{8}$$
$$\cos \delta \sin t = \sin z \sin A. \tag{9}$$

These equations are not adapted to logarithmic computations (unless addition and subtraction logarithmic tables are employed), and they will be further transformed.

Let m be a positive abstract quantity and M an angle such that

$$m \sin M = \sin z \cos A, \tag{10}$$
$$m \cos M = \cos z, \tag{11}$$

which is always possible [*Chauvenet's Plane Trig.* § 174]. Substituting these in (7), (8) and (9),

$$\sin \delta = m \sin (\phi - M),$$
$$\cos \delta \cos t = m \cos (\phi - M),$$
$$\cos \delta \sin t = \sin z \sin A.$$

From these and (10) and (11)

$$\tan M = \tan z \cos A, \tag{12}$$
$$\tan t = \frac{\tan A \sin M}{\cos (\phi - M)}, \tag{13}$$
$$\tan \delta = \tan (\phi - M) \cos t, \tag{14}$$

which completely effect the transformation. The computations are partially checked by (9).

The quadrant of M is determined by (10) and (11). t is greater or less than 180° according as A is greater or less than 180°, since both terminate on the same side of the meridian. The quadrant of δ is fixed by (14).

Example. At Ann Arbor, 1891, March 13, the altitude of *Regulus* is $+32^\circ\ 10'\ 15''.4$, and the azimuth is $283^\circ\ 5'\ 5''.6$. Find the declination and hour angle. [For instructions in the art of computing, see Appendix].

ϕ	$+42^\circ\ 16'\ 48''.0$, Naut. Alm., p. 482		
z	57 49 44 .6	$\tan(\phi - M)$	9.616914
A	283 5 6 .4	$\cos t$	9.728805
$\tan z$	0.201331	δ	$+12^\circ\ 29'\ 56''.4$
$\cos A$	9.354873		
M	$19^\circ\ 47'\ 40''.1$		Proof.
ϕ	42 16 48 .0	$\sin z$	9.927608
$\tan A$	0.633702_n	$\sin A$	9.988575_n
$\sin M$	9.529753	$\operatorname{cosec} t$	0.073401_n
$\sec(\phi - M)$	0.034339	$\cos \delta$	0.010417
$\tan t$	0.197794_n	$\log 1$	0.000001
t	$302^\circ\ 22'\ 54''.0$		
t	$20^h\ 9^m\ 31^s.6$.		

14. *Given the declination and hour angle of a star, required its azimuth and zenith distance.*

In the general equations (1), (2) and (3) let

$$b = 90^\circ - \delta, \quad c = 90^\circ - \phi, \quad A = t,$$
$$B = 180^\circ - A, \quad a = z;$$

and they become

$$\cos z = \sin\delta \sin\phi + \cos\delta \cos\phi \cos t, \tag{15}$$
$$\sin z \cos A = -\sin\delta \cos\phi + \cos\delta \sin\phi \cos t, \tag{16}$$
$$\sin z \sin A = \cos\delta \sin t, \tag{17}$$

To transform them for logarithmic computation, put

$$n \sin N = \sin\delta, \tag{18}$$
$$n \cos N = \cos\delta \cos t. \tag{19}$$

Whence

$$\tan N = \frac{\tan\delta}{\cos t}, \tag{20}$$

$$\tan A = \frac{\tan t \cos N}{\sin(\phi - N)}, \tag{21}$$

$$\tan z = \frac{\tan(\phi - N)}{\cos A}, \tag{22}$$

which effect the transformation. (17) furnishes a partial check on the computations.

Example. At Ann Arbor, 1891, March 13, when the hour angle of *Regulus* is $20^h\ 9^m\ 31^s.6$, what are the azimuth and zenith distance?

δ	$+\ 12°\ 29'\ 56''.4$, Naut, Alm., p. 332.		
t	302 22 54 .0		
$\tan\delta$	9.345719	$\tan(\phi - N)$	9.556204
$\cos t$	9.728805	$\cos A$	9.354873
N	$22°\ 29'\ 7''.9$	z	$57°\ 49'\ 44''.6$
ϕ	42 16 48 .0		Proof.
$\tan t$	0.197794_n	$\cos\delta$	9.989583
$\cos N$	9.965661	$\sin t$	9.926599_n
$\operatorname{cosec}(\phi - N)$	0.470247	$\operatorname{cosec} A$	0.011425_n
$\tan A$	0.633702_n	$\operatorname{cosec} z$	0.072392
A	$283°\ 5'\ 6''.4$	$\log 1$	9.999999

15. The angle POZ, Fig. 1, between the hour and vertical circles of a star, is called the star's *parallactic angle.* Let q represent it.

To find the parallactic angle when z, A and φ are given, we have, from (2) and (3),

$$\cos\delta\ \cos q = \sin z \sin\phi + \cos z \cos\phi \cos A, \tag{23}$$
$$\cos\delta \sin q = \sin A \cos\phi. \tag{24}$$

Assume

$$k \sin K = \sin\phi, \tag{26}$$
$$k \cos K = \cos\phi \cos A, \tag{27}$$

and we obtain

$$\tan K = \frac{\tan\phi}{\cos A}, \tag{28}$$

$$\tan q = \frac{\tan A \cos K}{\cos (K - z)}. \tag{29}$$

The quadrant of q is determined by (24) and (29).

16. To find the parallactic angle and zenith distance when δ, t and φ are given, we have, from (1), (2) and (3),

$$\cos z = \sin\delta \sin\phi + \cos\delta \cos\phi \cos t, \tag{30}$$
$$\sin z \cos\ q = \cos\delta \sin\phi - \sin\delta \cos\phi \cos t, \tag{31}$$
$$\sin z \sin\ q = \sin t \cos\phi. \tag{32}$$

Assume

$$l \sin L = \sin\phi, \tag{33}$$
$$l \cos L = \cos\phi \cos t, \tag{34}$$

and we obtain

$$\tan L = \frac{\tan \phi}{\cos t}, \tag{35}$$

$$\tan q = \frac{\tan t \cos L}{\sin (L - \delta)}, \tag{36}$$

$$\tan z = \frac{\tan (L - \delta)}{\cos q}, \tag{37}$$

The values of q obtained from the data of § 13 and § 14 are equal to each other, and to $312°25'\,33''.5$.

17. *Given the declination and zenith distance of a star, required its hour angle*

If a, b and c are the sides and A an angle of a spherical triangle, we have [*Chauvenet's Sph. Trig.* § 18]

$$\tan \tfrac{1}{2} \mathrm{A} = \pm \sqrt{\frac{\sin (\mathrm{s} - \mathrm{b}) \sin (\mathrm{s} - \mathrm{c})}{\sin \mathrm{s} \sin (\mathrm{s} - \mathrm{a})}},$$

in which $\mathrm{s} = \frac{1}{2}(\mathrm{a} + \mathrm{b} + \mathrm{c})$. If in this we substitute from triangle POZ

$$\mathrm{A} = t, \quad \mathrm{a} = z, \quad \mathrm{b} = 90° - \delta, \quad \mathrm{c} = 90° - \phi,$$

it reduces to

$$\tan \tfrac{1}{2} t = \pm \sqrt{\frac{\sin \frac{1}{2} [z + (\phi - \delta)] \sin \frac{1}{2} [z - (\phi - \delta)]}{\cos \frac{1}{2} [z + (\phi + \delta)] \cos \frac{1}{2} [z - (\phi + \delta)]}}. \tag{38}$$

To determine the quadrant of t it must appear from the data of the problem whether the star is west or east of the meridian. If it is west $\frac{1}{2} t$ is in the first quadrant; if east, $\frac{1}{2} t$ is in the second.

18. *Given the hour angle of a star, required its right ascension, and vice versa.*

In Fig. 1, for any star O we have

VM = right ascension of star $= a$,
MQ = hour angle of star $= t$,
VQ = sidereal time at observer's meridian $= \theta$.

Therefore

$$a = \theta - t, \tag{39}$$

and

$$t = \theta - a, \tag{40}$$

which effect the transformations.

19. *Given the right ascension and declination of a star, required its longitude and latitude, and vice versa.*

The transformations are effected by applying the general equations (1), (2) and (3) to the triangle POP', Fig. 1, in which

$$OP = 90^\circ - \delta, \quad OP' = 90^\circ - \beta, \quad OPP' = 90^\circ + \alpha,$$
$$OP'P = 90^\circ - \lambda, \quad PP' = \text{obliquity of ecliptic} = \varepsilon.$$

The value of ε is given in the Naut. Alm., p. 278.

20. *Given the right ascensions and declinations of two stars, required the distance between them.*

Let the co-ordinates of the stars be α', δ', and α'', δ'', and d the required distance. In the spherical triangle whose vertices are at the two stars and the pole, the sides are $90^\circ - \delta'$, $90^\circ - \delta''$ and d, and the angle at the pole is $\alpha'' - \alpha'$. Let B' represent the angle opposite $90^\circ - \delta'$. If in (1), (2) and (3) we put

$$\mathrm{a} = d, \quad \mathrm{B} = B', \quad \mathrm{b} = 90^\circ - \delta', \quad \mathrm{c} = 90^\circ - \delta'', \quad \mathrm{A} = \alpha'' - \alpha',$$

they become

$$\cos d = \sin \delta' \sin \delta'' + \cos \delta' \cos \delta'' \cos (\alpha'' - \alpha'), \tag{41}$$
$$\sin d \cos B' = \sin \delta' \cos \delta'' - \cos \delta' \sin \delta'' \cos (\alpha'' - \alpha'), \tag{42}$$
$$\sin d \sin B' = \cos \delta' \sin (\alpha'' - \alpha'). \tag{43}$$

If d can be determined from its cosine with sufficient precision, (41) gives the required distance; otherwise it should be determined from the tangent. If we assume

$$g \sin G = \cos \delta' \cos (\alpha'' - \alpha'), \tag{44}$$
$$g \cos G = \sin \delta', \tag{45}$$

we shall find that

$$\tan G = \cot \delta' \cos (\alpha'' - \alpha'), \tag{46}$$

$$\tan B' = \frac{\tan (\alpha'' - \alpha') \sin G}{\cos (\delta'' + G)}. \tag{47}$$

$$\tan d = \frac{\cot (\delta'' + G)}{\cos B'}. \tag{48}$$

(43) furnishes a partial check on the computations.

CHAPTER IV.

CORRECTION OF OBSERVATIONS.

21. The observed directions of all bodies in the solar system are sensibly different for observers at different places on the earth's surface. These differences must be allowed for before observations made at different places can be compared. This

is accomplished by reducing all observations to the center of the earth, to which point the Nautical Almanac ephemerides refer. A knowledge of the form and size of the earth is therefore indispensable.

FORM AND DIMENSIONS OF THE EARTH.

22. Geodetic measurements show that the earth is very nearly an oblate spheroid whose minor axis coincides with the polar axis.

Let $QPQ'P'$ be an elliptical section of the spheroid made by the meridian of an observer at O; A the center of the earth; NS the horizon; and let

a = semi-major axis of ellipse,
b = semi-minor axis of ellipse,
$\phi = OBQ$ = geographical latitude of O,
$\phi' = OAQ$ = *geocentric latitude* of O,
ρ = radius of earth at O,
$\phi' - \phi = AOB$ = reduction to geocentric latitude,
x, y = rectangular coordinates of O.

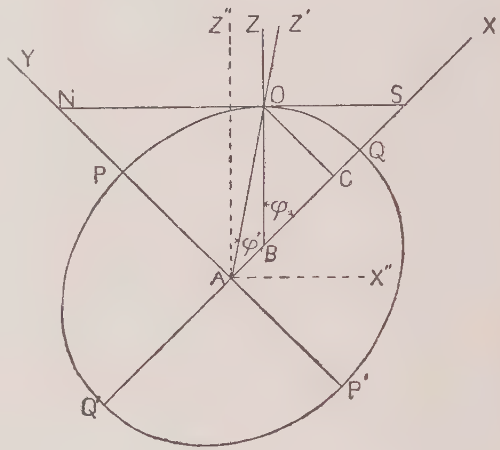

Fig. 2

23. *Given the geographical latitude, required the geocentric latitude.*

The equation of the ellipse (Fig. 2) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{49}$$

Differentiating, and substituting

$$\tan \phi = -\frac{dx}{dy}, \qquad \tan \phi' = \frac{y}{x},$$

we obtain the desired relation

$$\tan \phi' = \frac{b^2}{a^2} \tan \phi. \tag{50}$$

By discussing all available observations Bessel found

$$a = 3962.802 \text{ miles}, \quad b = 3949.555 \text{ miles};$$

whence

$$\log \frac{b^2}{a^2} = 9.9970915, \quad \text{eccentricity} = e = 0.0816967. \tag{51}$$

24. *To find the radius of the earth for a given latitude.* Substituting $x = \rho \cos \varphi'$ and $y = \rho \sin \varphi'$ in (49) and eliminating a and b by (50) we readily obtain

$$\rho = a \sqrt{\frac{\cos \phi}{\cos(\phi - \phi') \cos \phi'}}. \tag{52}$$

In using this equation make $a = 1$, since the equatorial radius is taken as the unit.

$\varphi' - \varphi$ can be expressed in terms of φ. If the equation

$$\tan x = p \tan y,$$

which is identical in form with (50), be developed in series it becomes [*Chauvenet's Plane Trig.* § 254]

$$x - y = q \sin 2y + \tfrac{1}{2} q^2 \sin 4y + \tfrac{1}{3} q^3 \sin 6y + \ldots,$$

in which

$$q = \frac{p-1}{p+1}.$$

Substituting from (50) the values corresponding to x, y and p, and dividing by $\sin 1''$ in order to express the result in seconds of arc, we obtain the practically rigorous formula

$$\phi' - \phi = -690''.65 \sin 2\phi + 1''.16 \sin 4\phi, \tag{53}$$

or

$$\phi' = \phi - 690''.65 \sin 2\phi + 1''.16 \sin 4\phi, \tag{54}$$

which determines φ' with greater accuracy than (50).

PARALLAX.

25. The *geocentric*, or *true*, *place* of a star is that in which it would be seen by an observer at the centre of the earth. The *apparent*,* or *observed*, *place* is that in which it is seen by the observer on the surface of the earth. The *parallax* of a star is

*The terms *true* and *apparent* are used in a relative sense only. In parallax the true place is the place corrected for parallax. In refraction the apparent place is affected by refraction, the true place is corrected for refraction; and similarly in other subjects.

2

the difference between its true and apparent places. It may also be defined as the angle at the star subtended by the radius of the earth drawn to the point of observation. This angle is a maximum for an observer at a given place when the star is seen in his horizon. It is then called the *horizontal parallax.* When the observer is at a place on the earth's equator this angle is called the *equatorial horizontal parallax.*

26. *To find the equatorial horizontal parallax of a star.*

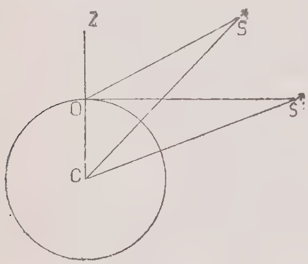

Fig. 3.

In Fig. 3 let S' be a star in the horizon of a point O on the earth's equator. Then if

a = equatorial radius of the earth = CO,
Δ = star's distance from earth's centre = CS',
π = equatorial horizontal parallax = $CS'O$,

we have

$$\sin \pi = \frac{a}{\Delta}. \tag{55}$$

The astronomical unit of distance is the mean distance of the earth from the sun. Using (55) with a and Δ expressed in terms of this unit. the Nautical Almanac tabulates the values of π for the moon [fourth page of the month] and the planets [pp. 218 249]; employing values of a and the earth's mean distance from the sun such that the sun's mean equatorial horizontal parallax is $8''.848$.

27. *To find the parallax in altitude or zenith distance, the earth being regarded as a sphere.*

In Fig. 3 let

z' = apparent zenith distance of a star S = ZOS,
z = true zenith distance = ZCS,
p = parallax in zenith distance = $CSO = z' - z$.

Then from the triangle COS

$$\frac{\sin p}{\sin z'} = \frac{a}{\Delta} = \sin \pi,$$

or

$$\sin p = \sin \pi \sin z'. \tag{56}$$

For all bodies except the moon we can write

$$p = \pi \sin z' = \pi \cos h'. \tag{57}$$

For refined observations (56) is not sufficiently exact, and recourse must be had to formulæ which consider the earth as a spheroid.

28. *Given the true zenith distance and azimuth of a star, required its apparent zenith distance and azimuth, the earth being regarded as a spheriod.*

Let the star be referred to a system of rectangular axes whose origin is at the point of observation, the positive axis of x being directed to the south point, the positive axis of y to the west point, and the positive axis of z to the zenith. Let

$$\begin{aligned} X', Y', Z' &= \text{the rectangular co-ordinates of the star,} \\ \Delta' &= \text{the star's distance from the observer,} \\ A' &= \text{its apparent azimuth,} \\ z' &= \text{its apparent zenith distance.} \end{aligned}$$

Then

$$\begin{aligned} X' &= \Delta' \sin z' \cos A', \\ Y' &= \Delta' \sin z' \sin A', \\ Z' &= \Delta' \cos z'. \end{aligned}$$

Again, let the star be referred to a second system of rectangular axes parallel to the first, the origin being at the center of the earth. Let

$$\begin{aligned} X, Y, Z &= \text{the rectangular co-ordinates of the star,} \\ \Delta &= \text{the star's distance from the origin,} \\ A &= \text{its true azimuth,} \\ z &= \text{its true zenith distance.} \end{aligned}$$

Then

$$\begin{aligned} X &= \Delta \sin z \cos A, \\ Y &= \Delta \sin z \sin A, \\ Z &= \Delta \cos z. \end{aligned}$$

Let the co-ordinates of the point of observation referred to the second system be X'', Y'', Z''. From Fig. 2 it is seen that

$$X'' = \rho \sin (\phi - \phi'), \quad Y'' = 0, \quad Z'' = \rho \cos (\phi - \phi').$$

Now

$$X' = X - X'', \quad Y' = Y - Y'', \quad Z' = Z - Z'',$$

and therefore

$$\left. \begin{aligned} \Delta' \sin z' \cos A' &= \Delta \sin z \cos A - \rho \sin (\phi - \phi'), \\ \Delta' \sin z' \sin A' &= \Delta \sin z \sin A, \\ \Delta' \cos z' &= \Delta \cos z - \rho \cos (\phi - \phi'). \end{aligned} \right\} \tag{58}$$

These equations completely determine Δ', z' and A', and therefore the parallax $z' - z$ and $A' - A$. It is better, how-

ever, to transform them so that the parallax can be computed directly. For this purpose, divide the equations through by Δ and put

$$f = \frac{\Delta'}{\Delta};$$

also substitute from (55), a being unity,

$$\sin \pi = \frac{1}{\Delta},$$

and we have

$$f \sin z' \cos A' = \sin z \cos A - \rho \sin \pi \sin (\phi - \phi'), \tag{59}$$
$$f \sin z' \sin A' = \sin z \sin A, \tag{60}$$
$$f \cos z' = \cos z - \rho \sin \pi \cos (\phi - \phi'). \tag{61}$$

From (59) and (60) we obtain

$$f \sin z' \sin (A' - A) = \rho \sin \pi \sin (\phi - \phi') \sin A, \tag{62}$$
$$f \sin z' \cos (A' - A) = \sin z - \rho \sin \pi \sin (\phi - \phi') \cos A. \tag{63}$$

Putting

$$m = \frac{\rho \sin \pi \sin (\phi - \phi')}{\sin z}, \tag{64}$$

(62) and (63) give

$$\tan (A' - A) = \frac{m \sin A}{1 - m \cos A}. \tag{65}$$

Multiplying (62) by $\sin \frac{1}{2} (A' - A)$, (63) by $\cos \frac{1}{2} (A' - A)$, adding the products and dividing by $\cos \frac{1}{2} (A' - A)$, we obtain

$$f \sin z' = \sin z - \rho \sin \pi \sin (\phi - \phi') \frac{\cos \frac{1}{2} (A' + A)}{\cos \frac{1}{2} (A' - A)}. \tag{66}$$

Let us assume

$$\tan \gamma = \tan (\phi - \phi') \frac{\cos \frac{1}{2} (A' + A)}{\cos \frac{1}{2} (A' - A)}; \tag{67}$$

then

$$f \sin z' = \sin z - \rho \sin \pi \cos (\phi - \phi') \tan \gamma. \tag{68}$$

This combined with (61) gives

$$f \sin (z' - z) = \rho \sin \pi \cos (\phi - \phi') \frac{\sin (z - \gamma)}{\cos \gamma}, \tag{69}$$

$$f \cos (z' - z) = 1 - \rho \sin \pi \cos (\phi - \phi') \frac{\cos (z - \gamma)}{\cos \gamma}. \tag{70}$$

Assume

$$n = \frac{\rho \sin \pi \cos (\phi - \phi')}{\cos \gamma}, \tag{71}$$

and we have

$$\tan (z' - z) = \frac{n \sin (z - \gamma)}{1 - n \cos (z - \gamma)}. \tag{72}$$

Formulæ (64) and (65) rigorously determine the parallax in azimuth, and (67), (71) and (72) the parallax in zenith distance. We may abbreviate the computation by writing (67) in the form

$$\gamma = (\phi - \phi') \cos A, \tag{73}$$

which is in all cases practically exact.

29. *Given the apparent zenith distance and azimuth, required the true zenith distance and azimuth, the earth being regarded as a spheroid.*

From (61) and (68) we can obtain

$$\sin (z' - z) = \frac{\rho \sin \pi \cos (\varphi - \varphi') \sin (z' - \gamma)}{\cos \gamma},$$

for which, since $\varphi - \varphi'$ and γ are small angles, we can write

$$\sin (z' - z) = \rho \sin \pi \sin (z' - \gamma), \tag{74}$$

in which γ is given without sensible error by

$$\gamma = (\phi - \phi') \cos A'. \tag{75}$$

We can obtain from (59) and (60)

$$\sin (A' - A) = \frac{\rho \sin \pi \sin (\phi - \phi') \sin A'}{\sin z}, \tag{76}$$

in which the value of z is found by the solution of (74). Formulæ (75), (74) and (76) completely solve the problem. For all bodies save the moon we may write

$$z' - z = \rho \pi \sin (z' - \gamma), \tag{77}$$

$$A' - A = \rho \pi \sin (\phi - \phi') \sin A' \operatorname{cosec} z'. \tag{78}$$

REFRACTION.

30. It is shown in Optics that when a ray of light passes obliquely from one transparent medium into another of greater density, it is refracted from its original direction according to the following laws:

(*a*). The incident ray, the normal to the surface which separates the two media at the point of incidence, and the refracted ray, lie in the same plane.

(*b*). The sines of the angles of incidence and refraction are inversely as the indices of refraction of the two media.

A ray of light coming from a star to an observer is assumed to travel in a straight line until it reaches the upper limit of the earth's atmosphere. It then passes continually from a rarer to a denser medium until it reaches the earth's surface. If we

regard the earth as a sphere, it follows from (a) and (b) that the path of the ray is a curve whose direction constantly approaches the center of the earth.

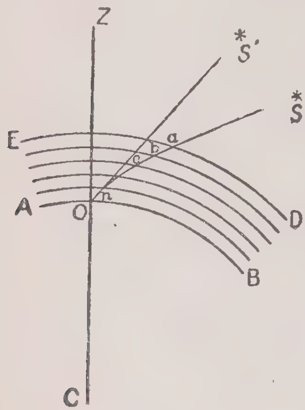

Fig. 4.

Let Fig. 4 represent a section of the earth and its atmosphere made by a vertical plane passing through the point of observation O and a star S. The path of the ray, $S\ a\ b\ldots n\ldots O$, lies wholly in this plane and is concave towards the earth. The *apparent* direction of the star is OS', a tangent to the curve at the point of observation. The *true* direction is that of a straight line joining O and S. The difference of these directions is the *refraction*. It appears that refraction increases the altitude (and decreases the zenith distance) of a star, but does not affect its azimuth.

The amount of the refraction depends upon the density of the air, which is a function of the atmospheric pressure and temperature. Our knowledge of the state of the atmosphere is very imperfect. The theory of refraction is complex and tedious, refraction tables to be reliable must be largely empirical, and we shall not attempt an investigation of the subject.

The *Pulcova Refraction Tables* given in the Appendix, TABLE I, are based on the formula

$$r = \mu \tan z\,(B\,T)^A\,\gamma^\lambda\,\sigma^i, \qquad (79)$$

in which z is the *apparent* zenith distance, μ, A, λ, and σ are functions of the apparent zenith distance, B depends on the reading of the *barometer*, T depends on the temperature of the column of mercury as indicated by the *attached* thermometer, γ depends on the temperature of the atmosphere as indicated by the *external* thermometer, i depends on the *time* of the year, and r is the refraction in seconds of arc.

For logarithmic computation (79) takes the form

$$\log r = \log \mu + \log \tan z + A\,(\log B + \log T) + \lambda \log \gamma + i \log \sigma. \qquad (80)$$

Observations should not be made at a greater zenith distance than 82° 30′, beyond which the amount of the refraction is uncer-

tain. We can compute an approximate value of the refraction, however, by means of the *Supplement* to TABLE I, which tabulates the values of $\log \mu \tan z$.

Example. Given the apparent zenith distance 81° 11′ 0″, Barom. 29.420 inches, Attached Therm. + 46°.5 F., External Therm. + 22°.3 F., time May 20, required the true zenith distance.

$\log B$	-0.00260	$\log \mu$	1.74132
$\log T$	-0.00055	$A \log B T$	9.99683
$\log B T$	-0.00315	$\lambda \log \gamma$	0.02456
A	1.0053	$i \log \sigma$	9.99995
$\log \gamma$	$+0.02337$	$\log \tan z$	0.80937
λ	1.0510	$\log r$	2.57203
$\log \sigma$	0.00026	r	6′ 13″.3
i	-0.21		

The true zenith distance is therefore 81° 17′ 13″.3.

If the true zenith distance is given and the apparent zenith distance is required, an approximate value of the latter is first found by applying the *mean refraction*, TABLE II, Appendix, to the true zenith distance, and then the refraction is given by (80) as before.

TABLE II is constructed from (80) for a mean state of the atmosphere, viz.: Barom. 29.5 inches, Att. Therm. 50°.0, and Ext. Therm. 50°.0. The factor σ^i is neglected.

In case no tables are available a very approximate value of the refraction is given by

$$r = \frac{983\, b}{460 + t} \tan z, \tag{81}$$

in which b is the barometer reading in inches, t the temperature of the atmosphere in degrees Fahr., and z the apparent zenith distance.* For zenith distances less than 75° it represents the Pulcova refractions within a second of arc, except for extreme states of the atmosphere. It is especially convenient for field work in which an aneroid barometer is used.

When the barometer and thermometer have not been read an approximate value of the refraction is given by

$$r = 57''.7 \tan z,$$

but for large zenith distances and extreme states of the atmosphere it cannot be used safely.

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* This formula is due to Professor Comstock. See *The Sidereal Messenger* for April, 1890.

REFRACTION IN RIGHT ASCENSION AND DECLINATION.

31. The change in zenith distance due to refraction gives rise to corresponding changes in right ascension and declination. We know the general relations existing between these co-ordinates, whence the relations existing between their increments may be found by differentiation. From (7), δ and z being the only variables, we have

$$\cos\delta\, d\delta = -(\sin z \sin\phi + \cos z \cos\phi \cos A)\, dz,$$

which reduces by means of (23) to

$$d\delta = -\cos q\, dz. \tag{82}$$

Differentiating (30), regarding z, δ and t as variables,

$$-\sin z\, dz = (\cos\delta \sin\phi - \sin\delta \cos\phi \cos t)\, d\delta - \cos\delta \cos\phi \sin t\, dt,$$

which by (31), (32) and (82) reduces to

$$\cos\delta\, dt = \sin q\, dz. \tag{83}$$

But from (40) $dt = -d\alpha$; and replacing dz by the refraction r, (82) and (83) become

$$d\delta = -r\cos q, \tag{84}$$
$$d\alpha = -r\sin q \sec\delta. \tag{85}$$

These corrections reduce from the apparent to the true values of α and δ. If the true place is given and the apparent place is required, the signs of the corrections must be reversed.

To compute r we must know z. If z and A are given, q is determined by (29); if t and δ are known, q and z are determined by (36) and (37).

DIP OF THE HORIZON.

32. At sea the altitudes of celestial objects are measured from the *visible* sea horizon. This is below the true horizon by an amount depending on the elevation of the observer's eye above the surface of the sea.

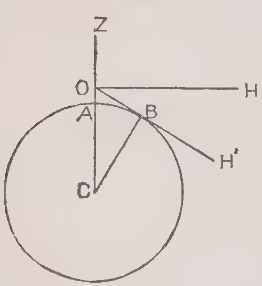

Fig. 5.

Let Fig. 5 represent a section of the earth made by a vertical plane passing through the eye of an observer at O. OH' is a line in the visible horizon, OH is the corresponding line in the true horizon, and HOH' is the *dip of the horizon.* Let

$x =$ the height of the eye above the water in feet $= OA$,
$a =$ the radius of the earth in feet $= AC$,
$D =$ the dip of the horizon $= HOH' = OCB$.

From geometry

$$\tan D = \frac{OB}{CB} = \frac{\sqrt{2ax + x^2}}{a} = \sqrt{\frac{2x}{a} + \frac{x^2}{a^2}} \,. \tag{86}$$

But $\frac{x^2}{a^2}$ is a very small quantity and may be neglected. $\tan D$ may be replaced by $D \tan 1''$. The apparent dip is affected by refraction. The amount of this refraction is uncertain, but an approximate value of the true dip is obtained by multiplying the apparent dip by the factor 0.9216. The mean value of a is 20888625 feet. Introducing these quantities in (86) it reduces to

$$D = 58''.82 \sqrt{x \text{ in feet}}, \tag{87}$$

by which amount the measured altitude must be decreased.

SEMIDIAMETER.

33. When we observe a celestial body having a well-defined disk, as in the case of the sun and moon, the measurements are made with reference to some point on the limb, and the position of the centre is obtained by correcting the observation for the angular semidiameter of the body.

The geocentric semidiameters of the sun, moon and major planets are tabulated in the Nautical Almanac. The apparent semidiameter of the moon, however, is appreciably different for different altitudes, on account of its nearness to the earth, and its value must be determined.

34. *To find the apparent semidiameter of the moon.*

Let Fig. 6 represent a section made by a plane passing through the observer O, the centre of the moon M, and the centre of the earth C, the earth being considered a sphere.* Let

$S =$ the moon's geocentric semidiameter, MCB;
$S' =$ the moon's apparent semidiameter, AOM;
$\Delta =$ the distance of the moon's centre from the earth's centre, CM;

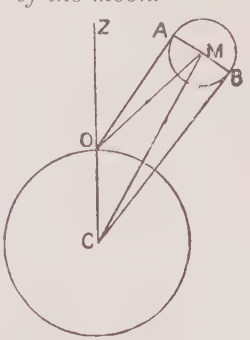

Fig. 6.

* The maximum error produced by neglecting the eccentricity of the meridian even in the case of the moon never exceeds $0''.06$.

Δ' = the distance of the moon's centre from the observer, OM;
π = the equatorial horizontal parallax of the moon;
p = the parallax in zenith distance, OMC;
z = the moon's true zenith distance, ZCM;
z' = the moon's apparent zenith distance, ZOM.

Then we can write

$$\frac{\sin S'}{\sin S} = \frac{\Delta}{\Delta'} = \frac{\sin (z + p)}{\sin z} = \cos p + \frac{\cos z \sin p}{\sin z}.$$

From § 27

$$\sin p = \sin \pi \sin z'; \tag{88}$$

therefore

$$\sin S' = \sin S \left(\cos p + \sin \pi \cos z \frac{\sin z'}{\sin z}\right). \tag{89}$$

(88) and (89) furnish very nearly an exact solution of the problem. For our purpose, and for all ordinary observations, we can write

$$S' = S (1 + \sin \pi \cos z). \tag{90}$$

35. *To find the contraction of any semidiameter of the sun or moon, produced by refraction.*

The apparent disk of the sun or moon is not circular, since the refraction for the lower limb is greater than for the centre, and that for the centre is greater than for the upper limb. It will be sufficiently exact to assume the disk to be an ellipse whose centre coincides with the centre of the sun or moon

The contraction of the vertical semidiameter is found by taking the difference of the refractions for the centre and the upper or lower limb.

The contraction of the horizontal semidiameter for all zenith distances less than 85° is *very nearly constant* and equal to about 0''.25. For our purpose it may be neglected and we shall not investigate the subject.

The contraction of any semidiamater making an angle q with the vertical semidiameter is readily obtained from the properties of the ellipse. Thus let

a = the horizontal semidiameter,
b = the vertical semidiameter,
S'' = the inclined semidiameter,

and we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
$$S'' \sin q = x,$$
$$S'' \cos q = y;$$

whence

$$S'' = \frac{ab}{\sqrt{a^2 \cos^2 q + b^2 \sin^2 q}} . \tag{91}$$

DIURNAL ABERRATION.

36. The observed direction of a star differs from its true direction in consequence of the motion of the observer in space. The ratio of the velocity of light to the velocity of the observer is finite, and a telescope changes its position appreciably while a ray of light is passing from the objective to the eyepiece.

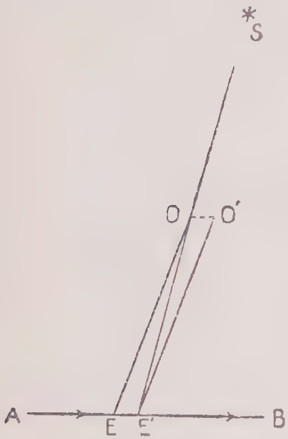

Fig. 7.

In Fig. 7 let O be the centre of the objective and E the center of the eyepiece of a telescope at the instant when a ray from a star S reaches the point O. If OE' represent the direction and velocity of the ray, and AB respresent the direction of the observer's motion and EE' his velocity, the telescope will be in the position $O'E'$ when the ray reaches E'. While the true direction of the star is $E'O$, the apparent direction is $E'O'$. The change in the apparent direction, $OE'O'$, is called the *aberration.* The star is displaced toward that point of the celestial sphere which the observer is momentarily approaching. To find the amount of this displacement let (Fig. 7)

$\gamma = BE'O$ = the angle between the true direction of the star and the line of the observer's motion,

$\gamma' = BE'O'$ = the angle between the apparent direction of the star and the line of the observer's motion,

$d\gamma = \gamma - \gamma'$ = the correction for aberration in the plane of the star and the observer's motion,

$V = OE'$ = the velocity of light,

$v = EE'$ = the velocity of the observer.

Then from the triangle EOE' we have

$$\sin (\gamma - \gamma') = \sin d\gamma = \frac{v}{V} \sin \gamma'.$$

But $d\gamma$ is always very small and we can write, without sensible error,

$$d\gamma = \frac{v}{V \sin 1''} \sin \gamma, \tag{92}$$

which determines the correction for aberration when v and V are known.

37. The velocity of the observer is made up of three parts: that due to the motion of the solar system as a whole, to the annual motion of the earth in its orbit, and to the diurnal rotation of the earth. The first need not be considered, since it affects the apparent place of a star by a constant quantity. The second gives rise to *annual aberration*, and will be considered in CHAP. V. The third gives rise to the *diurnal aberration.* This is a function of the observer's position on the earth, and will be treated as a correction to be applied to observed co-ordinates.

38. *To find the diurnal aberration in hour angle and declination.*

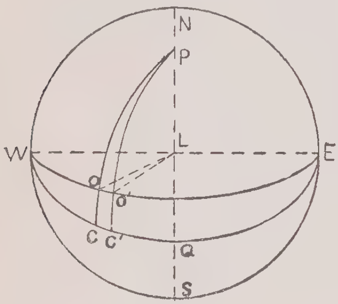

Fig. 8.

In Fig. 8 let $SENW$ be the horizon, EQW the equator, L the earth, O a star whose hour angle is t and declination δ, EOW a great circle through O and the east point of the horizon. Owing to the diurnal rotation of the earth the observer is moving directly toward the east point, and therefore the star's apparent place is shifted eastward in the plane EOW to some point O'. The aberration in this plane is OO', whose value is given by (92). It only remains to find the corresponding change in hour angle CC', and in declination $CO - C'O'$.

In the triangle ECO we have

$$CO = \delta, \qquad CE = 90° + t, \qquad ECO = 90°.$$

Now let

$$OE = \gamma, \qquad CEO = \omega, \qquad C'C = dt, \qquad CO - C'O' = d\delta,$$

and we can write

$$\sin \delta = \sin \gamma \sin \omega, \tag{93}$$
$$\sin t \cos \delta = - \cos \gamma, \tag{94}$$
$$\cos t \cos \delta = \sin \gamma \cos \omega. \tag{95}$$

(94) and (95) give by differentiation, t, δ and γ being variables,

$$- \sin t \sin \delta \, d\delta + \cos t \cos \delta \, dt = \sin \gamma \, d\gamma,$$
$$- \cos t \sin \delta \, d\delta - \sin t \cos \delta \, dt = \cos \gamma \cos \omega \, d\gamma.$$

Eliminating $d\delta$ and then dt, we obtain

$$\cos \delta \, dt = (\sin \gamma \cos t - \cos \gamma \sin t \cos \omega) \, d\gamma,$$
$$\sin \delta \, d\delta = - (\sin \gamma \sin t + \cos \gamma \cos t \cos \omega) \, d\gamma.$$

which, by means of (93), (94) and (95), reduce to

$$dt = \quad \cos t \sec \delta \frac{d\gamma}{\sin \gamma}, \tag{96}$$

$$d\delta = -\sin t \sin \delta \frac{d\gamma}{\sin \gamma}. \tag{97}$$

The value of the factor $\frac{d\gamma}{\sin \gamma}$ is given by (92) from the known values of v and V. For an observer at the earth's equator it is $0''.31$; in latitude φ it is $0''.31 \cos \varphi$. Substituting this value in (96) and (97) we obtain

$$dt = +0''.31 \cos \phi \cos t \sec \delta, \tag{98}$$

$$d\delta = -0\ \ .31 \cos \phi \sin t \sin \delta, \tag{99}$$

which are the corrections to be applied to the observed hour angle and declination.

When the star is observed on the meridian $t = 0$ and (98) and (99) become

$$dt = 0''.31 \cos \phi \sec \delta, \tag{100}$$

$$d\delta = 0\ \ . \tag{101}$$

39. *To find the diurnal aberration in azimuth and altitude.* The problem is identical with that in § 38 save that the horizon is the plane of reference, instead of the equator. If in (98) and (99) we replace t by A and δ by h we obtain the desired corrections,

$$dA = +0''.31 \cos \phi \cos A \sec h, \tag{102}$$

$$dh = -0\ \ .31 \cos \phi \sin A \sin h, \tag{103}$$

which are the corrections to be applied to the observed azimuth and altitude.

SEQUENCE AND DEGREE OF CORRECTIONS.

40. In applying the corrections considered in this chapter it is necessary that a proper sequence be followed.

In all cases the refraction must be applied first, its amount being obtained by the methods of § 30 and § 31.

Except in a few cases the diurnal aberration may be neglected.

Observations on the sun or moon refer to points on the limb. They must be reduced to the centre. In the case of the moon the reduction is made by formulæ (90) and (91); of the sun, by (91).

The parallax is now determined by the methods of § 27 or § 29. It is wholly inappreciable for the stars.

The degree of refinement to which these corrections should be carried, can be stated only in a general way. Usually it is

sufficient to compute them to one order of units lower than that to which the observations have been made. Thus, in reducing an observation made with a sextant reading to 10″, the corrections should be computed to the nearest second. If the mean of a large number of sextant readings is employed, it is advisable to carry the corrections to the tenth of a second; and similarly in other cases.

CHAPTER V.

PRECESSION.—NUTATION.—ANNUAL ABERRATION.—PROPER MOTION.*

41. While the relative positions of the fixed stars change very slowly,—and in most cases no change at all has been detected,—the *apparent* co-ordinates of all the stars are continually varying. These variations are divided into two general classes, secular and periodic.

Secular variations are very slow and nearly regular changes covering long periods of time, so that for a few years, and in some cases for centuries, they may be regarded as proportional to the time.

Periodic variations are changes which pass quickly from one extreme value to another, so that they cannot be treated as proportional to the time except for very short intervals.

The planes of the ecliptic and equator are subject to slow motions, which give rise to variations in the obliquity of the ecliptic and in the positions of the equinoxes. The co-ordinates of the stars therefore undergo changes which do not arise from the motions of the stars themselves, but from a shifting of the planes of reference and the origin of co-ordinates. The forces producing these changes are variable, and while the variations of the co-ordinates are progressive, they are not uniform. They may be regarded as made up of two parts, viz.: a secular variation called *precession*, and a periodic variation called *nutation*.

Owing to annual aberration [see § 37] the stars are not seen in their true positions, but are apparently displaced toward that point of the sphere which the earth is approaching, thus giving rise to periodic variations of their apparent co-ordinates.

In the case of stars having proper motions,—which are due

*In CHAPTER IV we considered the corrections to be applied to observations in order to reduce them to the centre of the earth. We now consider the corrections which affect the apparent geocentric co-ordinates of the stars. The observer is supposed to be at the centre of the earth.

to motions of the stars themselves, and to the motion of the solar system in space,—their positions on the sphere change, and give rise to secular variations of the co-ordinates.

42. In order that we may define the positions of the ecliptic and equator at any instant, it will be convenient to adopt the positions of these planes at some epoch as *fixed planes*, to which their positions at any other instant may be referred. Let their positions at the beginning of the year 1800 be adopted as the mean ecliptic and equator at that instant.

The *true equator* and *ecliptic* at any instant are the *real* equator and ecliptic at that instant. Their positions are affected by precession and nutation.

The positions of the *mean equator* and *ecliptic* at any instant are the positions these circles would occupy at that instant if they were affected by precession, but not by nutation.

The *mean place* of a star at any instant is its position referred to the mean equator and ecliptic of that instant. It is affected by precession and proper motion.

The *true place* of a star is its position referred to the true equator and ecliptic. It is the mean place plus the variation due to nutation.

The *apparent place* of a star is the position in which it would be seen by an observer (at the centre of the earth). It is the true place plus the variation due to annual aberration.

43. In solving the problems considered in the following chapters we require to know the *apparent* right ascensions and declinations of the celestial objects at the instants when they are observed. The apparent places of the sun, moon, major planets, and about three hundred of the brighter stars, are given in the Nautical Almanac at intervals such that their places for any instant may be obtained by interpolation. But occasionally it is desirable to employ stars not included in this list. If the mean places of these stars are given in the Nautical Almanac for the beginning of the year * they must be reduced, by means of the proper formulæ, to the apparent places at the times of observation. If we observe stars which are not contained in the Nautical Almanac we must refer for their positions to the general star catalogues which contain their mean places for the beginning of a certain year. These must be reduced to the corresponding mean places for the beginning of the year in

* This does not refer to the ordinary or tropical year, but to a *fictitious year*, which begins at the instant when the sun's mean longitude is 280°.

which the observations are made, and thence to the apparent places as before. We shall now very briefly consider the matters essential to these reductions.

PRECESSION.

44. If from the figure of the earth we subtract a sphere whose radius is equal to the earth's polar radius, there will remain a shell of matter symmetrically situated with reference to the equator. The attractions of the sun and moon on this shell tend to draw it into coincidence with the ecliptic. This tendency is resisted by the diurnal rotation of the earth. The combined effect of these forces is to shift the plane of the equator, without changing the obliquity of the ecliptic, in such a way that its intersection with the ecliptic continually moves to the west. This causes a common annual increase in the longitudes of the stars, which is called the *luni-solar precession.* It also affects the right ascensions and declinations, but not the latitudes.

The attractions of the other planets upon the earth tend to draw it out of the plane in which it is revolving. The effect is to shift the plane of the ecliptic in such a way that its intersection with the equator moves to the east. This causes a small annual decrease of the right ascensions of the stars, called the *planetary precession.* It also affects the latitudes and longitudes, but not the declinations.

The attractions of the planets produce a slight change in the obliquity of the ecliptic. Its annual effect upon the co-ordinates of the stars is combined with the luni-solar and planetary precession, the whole being called the *general precession.*

45. These motions are illustrated in Fig. 9. Let CV_0 be the fixed or mean ecliptic at the beginning of the year 1800, UV_0 the mean equator and V_0 the mean equinox. By the action of the sun and moon in the time t the equator is shifted to the position QV_1, the vernal equinox moves from V_0 to V_1, and V_0V_1 is the luni-solar precession in the interval t. By the attraction of the planets the ecliptic is shifted to the position CV_3, the vernal equinox moves from V_1 to V, and V_1V is the planetary precession in the interval t. Let

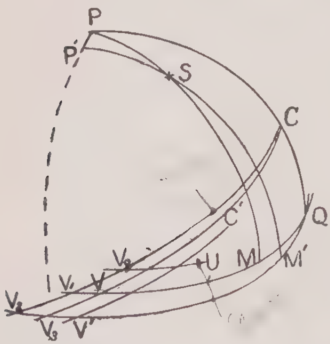

Fig. 9.

ε_0 = the mean obliquity of the ecliptic for 1800 = CV_0U,
ε_1 = the obliquity of the fixed ecliptic for $1800 + t = CV_1Q$,
ε = the mean obliquity of the ecliptic for $1800 + t = CVQ$,
ψ = the luni-solar precession in the interval $t = V_0V_1$,
ϑ = the planetary precession in the interval $t = V_1V$,
ψ_1 = the general precession in the interval $t = CV - CV_0$.

The values of these quantities are, according to Struve and Peters, for the beginning of the year 1800,

$$\left.\begin{aligned}
\varepsilon_0 &= 23^\circ\ 27'\ 54''.22,\\
\varepsilon_1 &= \varepsilon_0 + 0''.00000735\ t^2,\\
\varepsilon &= \varepsilon_0 - 0''.4738\ t - 0''.0000014\ t^2,\\
\psi &= 50''.3798\ t - 0''.0001084\ t^2,\\
\vartheta &= 0''.15119\ t - 0''.00024186\ t^2,\\
\psi_1 &= 50''.2411\ t + 0''.0001134\ t^2.
\end{aligned}\right\} \tag{104}$$

46. *Given the mean right ascension and declination* (α, δ) *of a star for any date* $1800 + t$, *required the mean right ascension and declination* (α', δ') *for any other date* $1800 + t'$.

In Fig. 9 let CV_2 be the ecliptic of 1800, V_1Q the mean equator of $1800 + t$, and V_2Q the mean equator of $1800 + t'$. If we denote by ascents the values given by (104) for the time t' we have

$$V_1V_2 = \psi' - \psi, \qquad QV_1V_2 = 180^\circ - \varepsilon_1, \qquad QV_2V_1 = \varepsilon_1'.$$

Now let

$$QV_1 = 90^\circ - z, \qquad QV_2 = 90^\circ + z', \qquad V_1QV_2 = \theta,$$

and we have [*Chauvenet's Sph. Trig.* §27]

$$\begin{aligned}
\cos \tfrac{1}{2}\theta \sin \tfrac{1}{2}(z' + z) &= \sin \tfrac{1}{2}(\psi' - \psi) \cos \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1),\\
\cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}(z' + z) &= \cos \tfrac{1}{2}(\psi' - \psi) \cos \tfrac{1}{2}(\varepsilon_1' - \varepsilon_1),\\
\sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}(z' - z) &= \cos \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\varepsilon_1' - \varepsilon_1),\\
\sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}(z' - z) &= \sin \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1).
\end{aligned}$$

But $\frac{1}{2}(z' - z)$ and $\frac{1}{2}(\varepsilon_1' - \varepsilon_1)$ are very small arcs, and we can write

$$\tan \tfrac{1}{2}(z' + z) = \tan \tfrac{1}{2}(\psi' - \psi) \cos \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1), \tag{105}$$

$$\tfrac{1}{2}(z' - z) = \frac{\tfrac{1}{2}(\varepsilon_1' - \varepsilon_1)}{\tan \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1)}, \tag{106}$$

$$\sin \tfrac{1}{2}\theta = \sin \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1), \tag{107}$$

which determine z', z and θ very accurately.

V and V' are the positions of the mean equinox for $1800 + t$ and $1800 + t'$. Representing the planetary precessions V_1V and V_2V' by ϑ and ϑ', we have

$$VQ = 90^\circ - z - \vartheta, \qquad V'Q = 90^\circ + z' - \vartheta';$$

3

and since for the star S we have $\alpha = VM$ and $\alpha' = V'M'$, we obtain

$$MQ = 90° - z - \vartheta - \alpha, \qquad M'Q = 90° + z' - \vartheta' - \alpha'.$$

Then if P and P' are the poles of the mean equator at $1800 + t$ and $1800 + t'$, and if we put

$$A = \alpha + \vartheta + z, \qquad A' = \alpha' + \vartheta' - z', \tag{108}$$

we have, in the triangle SPP',

$$PS = 90° - \delta, \quad P'S = 90° - \delta', \quad PP' = V_1QV_2 = \theta,$$
$$SPP' = 90° - MQ = A, \quad SP'P = 90° + M'Q = 180° - A'.$$

Substituting these in (2) and (3) we obtain

$$\cos\delta' \cos A' = \cos\delta \cos A \cos\theta - \sin\delta \sin\theta,$$
$$\cos\delta' \sin A' = \cos\delta \sin A,$$

from which we deduce

$$\cos\delta' \sin(A' - A) = \cos\delta \sin A \sin\theta (\tan\delta + \tan\tfrac{1}{2}\theta \cos A), \tag{109}$$
$$\cos\delta' \cos(A' - A) = \cos\delta - \cos\delta \cos A \sin\theta (\tan\delta + \tan\tfrac{1}{2}\theta \cos A); \tag{110}$$

or, putting

$$p = \sin\theta (\tan\delta + \tan\tfrac{1}{2}\theta \cos A), \tag{111}$$

we have

$$\tan(A' - A) = \frac{p \sin A}{1 - p \cos A}. \tag{112}$$

From the triangle SPP' we can also obtain [*Chauvenet's Sph. Trig.* §22]

$$\tan\tfrac{1}{2}(\delta' - \delta) = \tan\tfrac{1}{2}\theta \frac{\cos\frac{1}{2}(A' + A)}{\cos\frac{1}{2}(A' - A)}. \tag{113}$$

Having determined ε_1, ψ, ϑ, ε_1', ψ' and ϑ' from (104), z, z' and θ from (105), (106) and (107), and A from (108), we obtain α' from (111), (112) and (108), and δ' from (113).

Example. The mean place of *Polaris* for 1755.0 was

$$\alpha = 0^h\, 43^m\, 42^s.11, \quad \delta = +87°\, 59'\, 41''.11;$$

neglecting proper motion what what will be its mean place for 1900.0?

In this case $t = -45$ and $t' = +100$, and we find, from (104),

ε_1	23° 27′ 54″.23488	ε_1'	23° 27′ 54″.29350
ψ	— 37 47 .31	ψ'	+ 1 23 56 .90
ϑ	— 7 .29	ϑ'	+ 12 .70

and therefore

$\frac{1}{2}(\varepsilon_1' + \varepsilon_1)$	23° 27′ 54″.26
$\frac{1}{2}(\psi' - \psi)$	+ 1 0 52 .10
$\frac{1}{2}(\varepsilon_1' - \varepsilon_1)$	+ 0 .02931

$\tan\frac{1}{2}(\psi' - \psi)$	8.248163	$\sin A$	9.3125989
$\cos\frac{1}{2}(\varepsilon_1' + \varepsilon_1)$	9.962513	$\log p$	9.6050849
$\frac{1}{2}(z' + z)$	0° 55′ 50″.14	$\cos A$	9.9906400
$\log\frac{1}{2}(\varepsilon_1' - \varepsilon_1)$	8.467016	$\log p \cos A$	9.5957249
$\cot\frac{1}{2}(\psi' - \psi)$	1.751837	Sub*	0.2176761
$\operatorname{cosec}\frac{1}{2}(\varepsilon_1' + \varepsilon_1)$	0.399910	$\tan(A' - A)$	9.1353599
$\log\frac{1}{2}(z' - z)$	0.618763	$A' - A$	7° 46′ 36″.67
$\frac{1}{2}(z' - z)$	4″.16	A'	19 37 47 .01
z'	0° 55′ 54″.30	$a' = A' + z' - \vartheta'$	20 33 28 .61
z	0 55 45 .98	a'	$1^h\ 22^m\ 13^s$ .91
$\sin\frac{1}{2}(\psi' - \psi)$	8 248095		
$\sin\frac{1}{2}(\varepsilon_1' + \varepsilon_1)$	9.600090	$A' + A$	31° 28′ 57″.35
$\frac{1}{2}\theta$	0° 24′ 14″.16	$\tan\frac{1}{2}\theta$	7.8481943
a	10 55 31 .65	$\cos\frac{1}{2}(A' + A)$	9.9833991
$A = a + z + \vartheta$	11 51 10 .34	$\sec\frac{1}{2}(A' - A)$	0.0010009
$\tan\frac{1}{2}\theta$	7.848194	$\tan\frac{1}{2}(\delta' - \delta)$	7.8325943
$\cos A$	9.990640	$\frac{1}{2}(\delta' - \delta)$	0° 23′ 22″.85
$\tan\frac{1}{2}\theta \cos A$	7.838834	$\delta' - \delta$	0 46 45 .70
$\tan\delta$	1.4557773	δ'	88 46 26 .81
Add*	0.0001049		
$\sin\theta$	8.1492027		
$\log p$	9.6050849		

47. *Required the annual precession in right ascension and declination at any time* 1800 + t.

The precession for one year being small we can put, in (108) and (109), without sensible error,

$$\delta' = \delta, \quad \sin(A' - A) = (A' - A)\sin 1'', \quad \sin A = a \sin 1'', \quad \sin\theta \tan\tfrac{1}{2}\theta = 0,$$

and obtain

$$A' - A = a' - a + (\vartheta' - \vartheta) - (z' + z) = \theta \sin a \tan\delta. \qquad (114)$$

For (105) and (107) we may write

$$z' + z = (\psi' - \psi)\cos\varepsilon_1,$$
$$\theta = (\psi' - \psi)\sin\varepsilon_1.$$

Substituting these in (114) and dividing by $t' - t$ to obtain the annual precession, we have

* *Zech's Tafeln der Additions-und Subtractions-Logarithmen* are used here.

$$\frac{\alpha' - \alpha}{t' - t} = \frac{\psi' - \psi}{t' - t} \cos \varepsilon_1 - \frac{\vartheta' - \vartheta}{t' - t} + \frac{\psi' - \psi}{t' - t} \sin \varepsilon_1 \sin \alpha \tan \delta.$$

Similarly, from (113) we can obtain

$$\frac{\delta' - \delta}{t' - t} = \frac{\psi' - \psi}{t' - t} \sin \varepsilon_1 \cos \alpha.$$

In order to express the rate of change in α and δ at the instant $1800 + t$ we must let $t' - t$ become an infinitesimal. Passing to the limit we have

$$\frac{d\alpha}{dt} = \frac{d\psi}{dt} \cos \varepsilon_1 - \frac{d\vartheta}{dt} + \frac{d\psi}{dt} \sin \varepsilon_1 \sin \alpha \tan \delta,$$
$$\frac{d\delta}{dt} = \frac{d\psi}{dt} \sin \varepsilon_1 \cos \alpha\,.$$

If we put

$$m = \frac{d\psi}{dt} \cos \varepsilon_1 - \frac{d\vartheta}{dt}, \qquad n = \frac{d\psi}{dt} \sin \varepsilon_1.$$

we obtain for the annual precession at $1800 + t$

$$\frac{d\alpha}{dt} = m + n \sin \alpha \tan \delta\,, \tag{115}$$
$$\frac{d\delta}{dt} = n \cos \alpha\,. \tag{116}$$

From (104) we find

$$\frac{d\psi}{dt} \cos \varepsilon_1 = (50''.3798 - 0''.0002168\, t) \cos \varepsilon_1$$
$$= 46''.2135 - 0''.0001989\, t\,,$$
$$\frac{d\vartheta}{dt} = 0''.1512 - 0''.0004837\, t\,;$$

and therefore

$$m = 46''.0623 + 0''.0002849\, t\,, \tag{117}$$
$$n = 20''.0607 - 0''.0000863\, t\,. \tag{118}$$

Except for stars near the poles and for long intervals of time, formulæ (115) and (116) are very convenient for computing the whole precession between two dates. Thus if it is required to determine the precession in α and δ from $1800 + t$ to $1800 + t'$, we first obtain approximate values of α and δ for the middle date $1800 + \frac{1}{2}(t + t')$, then compute the annual precession for this date, which is very approximately the *average* annual precession for the interval $t' - t$, and thence the whole precession by multiplying this by $t' - t$.

It is convenient to have the values of m and n given by (117) and (118) tabulated as follows:

Date.	$\frac{1}{15}m$.	$\log \frac{1}{15}n$.	$\log n$.
1750	$3^s.06987$	0.126348	1.302439
1760	3 .07006	0.126330	1.302421
1770	3 .07025	0.126311	1.302402
1780	3 .07044	0.126292	1.302383
1790	3 .07063	0.126274	1.302365
1800	3 .07082	0.126255	1.302346
1810	3 .07101	0.126236	1.302327
1820	3 .07120	0.126218	1.302309
1830	3 .07139	0.126199	1.302290
1840	3 .07158	0.126180	1.302271
1850	3 .07177	0.126162	1.302253
1860	3 .07196	0.126143	1.302234
1870	3 .07215	0.126124	1.302215
1880	3 .07234	0.126106	1.302197
1890	3 .07253	0.126087	1.302178
1900	3 .07272	0.126068	1.302159

Example. The mean place of β *Orionis* for 1850.0 was

$$\alpha = 5^h\, 7^m\, 19^s.856, \qquad \delta = -8^\circ\, 22'\, 44''.74;$$

neglecting proper motion find its mean place for 1900.0.

Using the values of m and n for the middle date 1875.0, and α and δ for 1850.0, we obtain very nearly the annual precession in α and δ for 1862.5 from (115) and (116).

$\log \frac{1}{15} n$	0 .126115		
$\sin \alpha$	9 .988429		
$\tan \delta$	9.168186_n		
log	9.282730_n		
no	$-0^s.19175$	$\log n$	1.302206
$\frac{1}{15} m$	3 .07224	$\cos \alpha$	9.357543
$\frac{d\alpha}{dt}$	2 .88049	$\frac{d\delta}{dt}$	$4''.568$

The approximate co-ordinates of the star for 1875.0 are

$$\alpha = 5^h\, 8^m\, 31^s 87, \quad \delta = -8^\circ\, 20'\, 50''.5.$$

Using these values we have

$\log \frac{1}{15} n$	0 .126115		
$\sin \alpha$	9 .988955		
$\tan \delta$	9.166514_n		
log	9.281584_n		
no	$-0^s.19124$	$\log n$	1.302206
$\frac{1}{15} m$	3 .07224	$\cos \alpha$	9.347705
$\frac{d\alpha}{dt}$	2 .88100	$\frac{d\delta}{dt}$	$4''.46592$

These are the very approximate values of the annual precession for 1875.0, and the mean place for 1900.0 is therefore

$$\alpha' = 5^h\ 9^m\ 43^s.906, \qquad \delta' = -\ 8^\circ\ 19'\ 1''.44,$$

which is practically identical with that given by the rigorous method of § 46.

In many star catalogues the annual precession in α and δ is given for each star for the epoch of the catalogue, by means of which the approximate place of the star for the middle time is found at once, and the first approximation made above is avoided.

PROPER MOTION.

48. The proper motion of a star has already been defined to be a motion of the star itself on the surface of the sphere. It is assumed to take place in the arc of a great circle, and to be uniform. The proper motions in right ascension and declination are the components of this motion in and perpendicular to the equator. They are variable since the equator is a moving circle, and it must be specified to what equator they refer.

When a star's place is required to be very accurately known, its position should be taken from as many catalogues as possible. In order that the data thus obtained may be properly combined, a thorough knowledge of the subject of proper motion is essential.

49. *Given the observed mean places* (α, δ) *of a star for* $1800 + t$ *and* (α', δ') *for* $1800 + t'$, *required the annual proper motion.*

Starting from the first observed place and computing the precession for the interval $t' - t$ by the methods § 46 or § 47, let the resulting place for $1800 + t'$ be α_1, δ_1. The discrepancies $\alpha' - \alpha_1$ and $\delta' - \delta_1$ are due to proper motion, and the annual proper motion for the interval is

$$d\alpha' = \frac{\alpha' - \alpha_1}{t' - t}, \quad d\delta' = \frac{\delta' - \delta_1}{t' - t}, \tag{119}$$

referred to the equator of $1800 + t'$. Starting from the second observed place, computing the precession for the interval $t - t'$, and applying it to α' and δ', let the resulting place for $1800 + t$ be α_2, δ_2. The annual proper motion for the interval is

$$d\alpha = \frac{\alpha - \alpha_2}{t - t'}, \quad d\delta = \frac{\delta - \delta_2}{t - t'}, \tag{120}$$

referred to the equator of $1800 + t$.

Example. The mean places of *Polaris* for 1755.0 and 1900.0 given in *Newcomb's Standard Stars* are

$$\begin{array}{llll} \text{for } 1755.0, & \alpha = 0^h\ 43^m\ 42^s.11, & \delta = +\,87^\circ\ 59'\ 41''.11, \\ \text{for } 1900.0, & \alpha' = 1\ \ 22\ \ 33\ .76, & \delta' = +\,88\ \ 46\ \ 26\ \ .66; \end{array}$$

determine the proper motion referred to the equator of 1900.0.

By applying the precession to the place for 1755.0 the place for 1900.0 was found to be, § 46,

$$\alpha_1 = 1^h\ 22^m\ 13^s.91, \qquad \delta_1 = +\,88^\circ\ 46'\ 26''.81,$$

and therefore, by (119), the annual proper motion of *Polaris* referred to the equator of 1900.0 is

$$d\alpha' = +\,0^s.1369, \qquad d\delta' = -\,0''.00103.$$

50. *Given the proper motion* ($d\alpha$, $d\delta$) *referred to the equator of* 1800 + t, *required the corresponding proper motion* ($d\alpha'$, $d\delta'$) *referred to the equator of* 1800 + t', *and vice versa.*

When the star S (Fig. 9) moves on the surface of the sphere it causes variations in all the parts of the triangle $SP'P$, except $P'P$. The solution of the present problem requires a knowledge of the relations existing between these variations.

If in a spherical triangle ABC we suppose all the parts except a to vary, we can write [*Chauvenet's Sph. Trig.*, § 153, (286) *and* (287)]

$$\begin{aligned} \sin \mathrm{c}\, d\mathrm{B} &= \sin \mathrm{A}\, d\mathrm{b} - \sin \mathrm{a} \cos \mathrm{A} \sin \mathrm{B} \operatorname{cosec} \mathrm{A}\, d\mathrm{C}, \\ d\mathrm{c} &= \cos \mathrm{A}\, d\mathrm{b} + \sin \mathrm{a} \sin \mathrm{B}\, d\mathrm{C}. \end{aligned}$$

Substituting in these, from § 46,

$$\begin{gathered} \mathrm{a} = PP' = \theta, \quad d\mathrm{b} = d\,(SP) = -\,d\delta, \quad d\mathrm{c} = d\,(SP') = -\,d\delta', \\ d\mathrm{B} = d\,(SP'P) = d\,(180^\circ - A') = -\,d\alpha', \quad d\mathrm{C} = d\,(SPP') = dA = d\alpha, \end{gathered}$$

and putting γ for A, we obtain

$$\cos \delta'\, d\alpha' = \sin \gamma\, d\delta + \cos \delta \cos \gamma\, d\alpha, \tag{121}$$

$$d\delta' = \cos \gamma\, d\delta - \cos \delta \sin \gamma\, d\alpha, \tag{122}$$

in which γ is determined by

$$\sin \gamma = \sin \theta \sin A \sec \delta' = \sin \theta \sin A' \sec \delta, \tag{123}$$

$$\cos \gamma = (\cos \theta - \sin \delta \sin \delta') \sec \delta \sec \delta'. \tag{124}$$

These determine the proper motion for 1800 + t' in terms of that for 1800 + t.

If in (121) and (122) we eliminate $d\delta$ and then $d\alpha$, we obtain

$$\cos\delta\, d\alpha = \cos\delta' \cos\gamma\, d\alpha' - \sin\gamma\, d\delta', \qquad (125)$$
$$d\delta = \cos\delta' \sin\gamma\, d\alpha' + \cos\gamma\, d\delta', \qquad (126)$$

which determine the proper motion for $1800 + t$ in terms of that for $1800 + t'$.

Example. The proper motion of *Polaris* referred to the equator of 1900.0 is

$$d\alpha' = + 0^s.1369 = + 2''.0535, \quad d\delta' = - 0''.00103.$$

Deduce the proper motion referred to the equator of 1755.0.

Using θ and A from § 46 and $\delta' = + 88° 46' 26''.66$ we find, from (123),

$$\sin\gamma = 9.131493, \quad \cos\gamma = 9.995984,$$

and therefore from (125) and (126) we obtain

$$d\alpha = + 1''.2480 = + 0^s.0832, \quad d\delta = + 0''.00493.$$

51. *Given the proper motion* ($d\alpha$, $d\delta$) *and the mean place* (α, δ) *of a star for the epoch* $1800 + t$, *required its mean place* (α', δ') *for the epoch* $1800 + t'$.

The proper motion for the whole interval $t' - t$ is first computed and applied to the mean place for $1800 + t$. With the resulting values of α and δ, which we shall denote by α_1 and δ_1, the precession is computed and applied to α_1 and δ_1. The result is the star's mean place for $1800 + t'$.

If the proper motion ($d\alpha'$, $d\delta'$) is given for the epoch $1800 + t'$, we first compute the precession, using α and δ, and then apply the proper motion for the interval.

Example 1. Given the mean place and proper motion of *Polaris* for 1755.0,

$$\alpha = 0^h\, 43^m\, 42^s.11, \quad \delta = + 87°\; 59'\; 41''.11,$$
$$d\alpha = + 0^s.0832, \quad d\delta = + 0''.00493,$$

required the mean place for 1900.0.

The proper motion for the interval is

$$+ 0^s.0832 \times 145 = + 12^s.06, \quad + 0''.00493 \times 145 = + 0''.71.$$

Therefore

$$\alpha_1 = 0^h\, 43^m\, 54^s.17, \quad \delta_1 = + 87°\; 59'\; 41''.82.$$

Employing these values in § 46 we find for 1900.0

$$\alpha' = 1^h\, 22^m\, 33^s.76, \qquad \delta' = +\, 88^\circ\, 46'\, 26''.66.$$

Example 2. The proper motion of β *Orionis* referred to the equator of 1900.0 is

$$d\alpha = -\, 0^s.00027, \qquad d\delta = -\, 0''.0061.$$

Include this in the example of § 47.

The proper motion for the interval is

$$-\, 0^s.00027 \times 50 = -\, 0^s.013, \qquad -\, 0''.0061 \times 50 = -\, 0''.30;$$

and therefore the mean place for 1900.0 is

$$\alpha' = 5^h\, 9^m\, 43^s.893, \qquad \delta' = -\, 8^\circ\, 19'\, 1''.74.$$

52. It will be seen from § 47 that the annual precession is a slowly varying quantity. The change in its value in one hundred years is called the *secular variation* of the precession. Many star catalogues give not only the mean place and annual precession of a star but also the secular variation and proper motion. In this case the reduction of the mean place of a star from the epoch of the catalogue $1800 + t$ to that for $1800 + t'$ is readily made. For if

$p =$ the annual precession for the epoch $1800 + t$,
$\Delta p =$ the secular variation,
$\mu =$ the proper motion,

the reduction for the interval $t' - t$ will be the annual change for the middle time multiplied by $t' - t$, or

$$\left[p + \frac{\Delta p}{100} \cdot \frac{t' - t}{2} + \mu \right] (t' - t), \tag{127}$$

which form applies both to the right ascension and the declination.

Example. *Newcomb's Standard Stars* gives the following data for β *Orionis* for the epoch 1850.0:

$$\alpha = 5^h\, 7^m\, 19^s.856, \quad p = +\, 2^s.87999, \quad \Delta p = +\, 0^s.00400, \quad \mu = -\, 0^s.00025,$$
$$\delta = -\, 8^\circ\, 22'\, 44''.74, \quad p' = +\, 4''.5687, \quad \Delta p' = -\, 0''.4109, \quad \mu' = -\, 0''.0061;$$

required the mean place for 1900.0.

Substituting those values in (127) we obtain the reductions for the interval,

$$\text{for } \alpha, +\, 144^s.038, \qquad \text{for } \delta, +\, 222''.99,$$

and the mean place for 1900.0 is, therefore,

$$\alpha' = 5^h\ 9^m\ 43^s.894, \qquad \delta' = -8°\ 19'\ 1''.75.$$

REDUCTION TO APPARENT PLACE.

53. The mean place of a star for the beginning of the required year having been obtained by any of the above methods, it remains to determine its apparent place for any given instant. Thus if we desire the apparent place for a time τ from the beginning of the year, we obtain the mean place by adding the precession and proper motion for the interval τ, then the true place by adding the nutation, and finally the apparent place by adding the annual aberration. The reduction to the mean place could be performed as before; we could determine the nutation by evalutating the long and tedious nutation formulæ, the deduction of which belongs to physical astronomy; and we could obtain the annual aberration from equations which we could deduce by methods analogous to those of § 38. But this process is exceedingly laborious. By judiciously combining the terms of the various formulæ involved in the reductions, Bessel was able to propose two simple and closely related methods, which are now in common use.

54. *Given the mean place* (α, δ) *of a star for the beginning of the year, required its apparent place* (α', δ') *for any instant* τ.*

(a). The reduction is made by the formulæ

$$\alpha' = \alpha + \tau\mu + Aa + Bb + Cc + Dd + \tfrac{1}{15}E, \tag{128}$$
$$\delta' = \delta + \tau\mu' + Aa' + Bb' + Cc' + Dd', \tag{129}$$

in which $\tau\mu$ and $\tau\mu'$ represent the proper motion and Aa and Aa' the precession in the interval τ, $Bb + \frac{1}{15}E$ and Bb' the nutation, and $Cc + Dd$ and $Cc' + Dd'$ the annual aberration at the instant τ. A, B, C, D and E are the *Besselian star-numbers.* They are functions of the time. The Nautical Almanac gives their general values on p. 280, and tabulates the logarithms of A, B, C and D for every day of the year on pp. 281–284. The value of E is given in the same place. It is a slowly varying quantity whose value never exceeds $0''.05$ and can generally be neglected.

a, b, c, d, a', b', c' and d' are *Bessel's star-constants.* They are functions of the star's place and the obliquity of the ecliptic, and are defined by the equations

* See § 43, footnote, and the Nautical Almanac, p. 280.

$$\left.\begin{array}{ll} a = \frac{1}{15} m + \frac{1}{15} n \sin \alpha \tan \delta, & a' = n \cos \alpha, \\ b = \frac{1}{15} \cos \alpha \tan \delta, & b' = - \sin \alpha, \\ c = \frac{1}{15} \cos \alpha \sec \delta, & c' = \tan \varepsilon \cos \delta - \sin \alpha \sin \delta, \\ d = \frac{1}{15} \sin \alpha \sec \delta, & d' = \cos \alpha \sin \delta. \end{array}\right\} \quad (130)$$

In some star catalogues the logarithms of the star-constants are given for each star. But these values become obsolete in a few years, and must be computed anew from (130), since m, n, α, δ and ε are slowly varying quantities.

Example. Required the apparent place of 38 *Lyncis* for the upper transit at Ann Arbor, 1891 March 16.

From the *Berliner Jahrbuch*, p. 180, star 135, we find for 1891.0,

$$\alpha = 9^h\ 12^m\ 3^s.671, \qquad \delta = +37^\circ\ 15'\ 48''.43,$$
$$\mu = \quad -0.0030, \qquad \mu' = \quad -0\ .114.$$

The upper transit occurs therefore at Washington sid. time $9^h\ 39^m$, or $1^h\ 53^m$ before mean midnight. Taking the values of log A, etc., from the Nautical Almanac, p. 281, for this instant, and the values of log a, etc., from the *Jahrbuch*, p. 329, star 135, the computation is conveniently arranged as below.

log a 0.5744	log b 8.5764_n	log c 8.7943_n	log d 8.7485
log A 9.0221_n	log B 0.5804_n	log C 1.2722_n	log D 0.1239
log a' 1.1734_n	log b' 9.8254_n	log c' 8.7763_n	log d' 9.6533_n

α	$9^h\ 12^m\ 3^s.671$	δ	$+37^\circ\ 15'\ 48''.43$
$\tau\mu$	$-\ 0.001$	$\tau\mu'$	$-\ 0\ .02$
Aa	$-\ 0.395$	Aa'	$+\ 1\ .57$
Bb	$+\ 0.143$	Bb'	$+\ 2\ .55$
Cc	$+\ 1.165$	Cc'	$+\ 1\ .12$
Dd	$+\ 0.075$	Dd'	$-\ 0\ .60$
$\frac{1}{15}E$	$-\ 0.003$		
α'	9 12 4.655	δ'	$+37$ 15 53 .05

This method of reduction should be employed when the star-constants are given in the catalogues with sufficient accuracy, or when the apparent places of the same star are required for several dates.

In using the data of reduction furnished by the English and French annuals and catalogues, the computer must be careful to follow their formulæ; for while the *form* of reduction usually agrees with the American and German form, the *notation* is different, A and B in the one corresponding to C and D respectively in the other. This applies also to the American Nautical Almanac previous to 1865.

(*b*). When the catalogues do not give the values of log a, log b, etc., and when only one or two places of the same star are desired, another form of reduction is preferable. If we put

$$f = \tfrac{1}{15}\,m\,A + \tfrac{1}{15}\,E, \qquad h \sin H = C,$$
$$g \sin G = B, \qquad h \cos H = D,$$
$$g \cos G = n\,A, \qquad i = C \tan \varepsilon,$$

the formulæ (128) and (129) become

$$\alpha' = \alpha + \tau\mu + f + \tfrac{1}{15}\,g \sin(G + \alpha) \tan\delta + \tfrac{1}{15}\,h \sin(H + \alpha) \sec\delta, \tag{131}$$
$$\delta' = \delta + \tau\mu' + g \cos(G + \alpha) + h \cos(H + \alpha) \sin\delta + i \cos\delta, \tag{132}$$

in which the terms involving f, g and G denote the precession and nutation, and the terms involving h, H and i, the annual aberration. These auxiliary quantities are called the *independent star-numbers*. The values of τ, f, G, H, log g, log h and log i are given in the Nautical Almanac, pp. 285-292, for every day of the year.

Example. Required the apparent place of 38 *Lyncis* for the upper transit at Ann Arbor, 1891 March 16.

Using the data given above, the computation is conveniently made as below.

G	240° 59′	log g	0.6387	α	$9^h\ 12^m\ 3^s.671$
α	138 1	cos $(G + \alpha)$	9.9757	$\tau\mu$	— 0 .001
H	274 3			f	— 0 .326
		log h	1.2733	(1)	+ 0 .072
log $\frac{1}{15}$	8.8239	cos $(H + \alpha)$	9.7887	(2)	+ 1 .240
log g	0.6387	sin δ	9.7821	α'	9 12 4 .656
sin $(G + \alpha)$	9.5126	log (4)	0.8441		
tan δ	9.8813			δ	+ 37° 15′ 48″.43
log (1)	8.8565	log i	0.9096_n	$\tau\mu'$	— 0 .02
		cos δ	9.9008	(3)	+ 4 .12
log $\frac{1}{15}$	8.8239			(4)	+ 6 .98
log h	1.2733			(5)	— 6 .46
sin $(H + \alpha)$	9.8969			δ'	+ 37 15 53 .05
sec δ	0.0992				
log (2)	0.0933				

CHAPTER VI.

ANGLE AND TIME MEASUREMENT.

55. The degree of refinement to which an observer can carry the determination of his geographical position and the time depends upon the accuracy attainable in *pointing the telescope*, in *reading the angle* corresponding to the pointing, and in

noting the time when the pointing is made. In general, these elements are of equal importance. For any given telescope the first depends upon the observer's skill. In the last two the observer's skill is assisted by various mechanical devices.

THE VERNIER.

56. An angle is usually measured by means of a graduated *circle*, or arc, whose centre is at the vertex of the angle. Closely fitting upon the graduated arc of the circle and centred with it is another graduated arc called the *vernier*, which is so arranged upon an arm that it moves with reference to the circle when the telescope moves. The angle to be read is that included between the zero line of the circle and the zero line of the vernier. The zero of the vernier generally falls between two consecutive lines on the circle. The angle corresponding to the whole divisions can be read off at once; it is the object of the vernier to determine the fractional part of a division. It is so constructed that n of its divisions are equal in length to $n - 1$ divisions of the circle. If we let

d = the value of one division of the circle,
d' = the value of one division of the vernier,

we have

$$(n - 1)\, d = nd',$$

or

$$d - d' = \frac{d}{n}. \tag{133}$$

$d - d'$ is called the *least reading* of the vernier. If now the zero of the vernier coincides with a division line of the circle, the circle reading gives the required angle at once. If the first vernier line coincides with a circle line, the zero of the vernier is $d - d'$ beyond a line of the circle, and the circle reading must be increased by the least reading. If the second vernier line is in coincidence with a circle line the circle reading must be increased by twice the least reading, etc. For example, the value of a division of a sextant is $10'$, and 60 divisions of the vernier correspond in length to 59 divisions of the circle. The least reading is $10' \div 60 = 10''$. In measuring a certain angle the zero of the vernier fell between $42^\circ\ 40'$ and $42^\circ\ 50'$, and the 26th line of the vernier coincided with a circle line. The required reading was $42^\circ\ 40' + 26 \times 10'' = 42^\circ\ 44'\ 20''$. In

practice no computation is necessary, the number of minutes being read directly from the numbers on the vernier.

THE READING MICROSCOPE.

37. In very fine instruments the vernier is replaced by a *reading microscope*, the optical axis of which is perpendicular to the plane of the divided circle. The microscope is so adjusted that an image of the circle divisions is formed in the common focus of the objective and ocular. In the same focus are two very fine *micrometer wires* (usually spider-lines, which either intersect or are parallel and close together). They are stretched upon a light frame whose plane is parallel to the plane of the circle, and which is moved in the direction of a tangent to the circle by turning a fine *micrometer screw.* Fixed upon the projecting end of the screw is a cylindrical *micrometer head.* This is graduated into either 60 or 100 parts, and is used for reading the fractional parts of a revolution of the screw, the readings being made with reference to a fixed index. The whole number of revolutions is indicated by a scale either inside or outside of the microscope.

Let the micrometer screw be turned until the wires are in the centre of the field of view and the reading of the head is zero. The position now occupied by the wires is the fixed point of reference. The angle to be read is that included between the zero of the circle and this point. If now the micrometer wires coincide with a line of the circle the desired reading is obtained at once. If they fall beyond a certain line the fractional part of a division is determined by moving the wires from the point of reference into coincidence with the line. The distance passed over is determined from the micrometer reading and the known angular value of one revolution of the screw. In setting the wires upon any circle division the last motion of the micrometer should always take place in the same direction, any lost or dead motion being thereby avoided.

When the microscope is properly adjusted, a whole number of revolutions of the screw corresponds exactly to the distance between two consecutive circle lines. But this adjustment once made does not remain, owing to changes of temperature, etc. It is customary to determine from time to time the error which arises and allow for it. This is called the *error of runs.* Its value is found by measuring several divisions in different parts

of the circle, and taking the mean of the measures in order to eliminate as far as possible any errors in the graduations.

To illustrate, let a circle be graduated to $5'$, let the value of a revolution of the screw be $1'$, and let the head be divided into 60 parts. To find the error of runs—which is equivalent to finding the value of a revolution of the screw—let the mean of the measures of ten divisions of the circle be 4 revolutions and 56.4 divisions of the head. The correction for runs per minute is $+ 3''.6 \div 5 = + 0''.72$. Let an angle be read such that the circle reading is $62° \; 15'$, and the micrometer reading is 2 rev. 15.9 div. The correction for runs is $+ 1''.6$, and the angle is, therefore, $62° \; 17' \; 17''.5$.

ECCENTRICITY.

58. The centre of the arm which carries the vernier or microscope never coincides exactly with the centre of the circle, and an error due to this eccentricity enters into the circle readings. In Fig. 10 let C' be the centre of the vernier, C the centre of the circle, D the point of intersection of the circle DAB and the line CC' produced, A the zero point of the circle, and M the position of the vernier or microscope. The pointing of the telescope corresponds to the direction $C'M$ while the circle reading refers to the direction CM. The correction for eccentricity is therefore $C'MC$. To find its value let

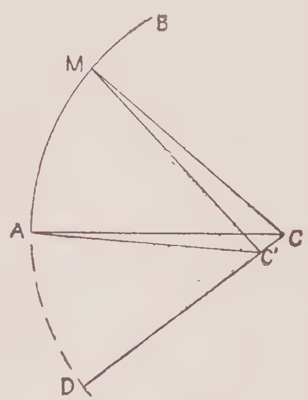

Fig. 10.

CC' = the eccentricity = e,
ε = the correction for eccentricity = $C'MC$,
M = the observed reading of the circle,
R = the true reading of the circle = $M + \varepsilon$,
r = the radius of the circle,
α = the angle $DC'A$,
β = the angle $AC'M$.

From the triangle $C'MC$ we can write

$$r \sin \varepsilon = e \sin (\beta + \alpha).$$

Since r is the unit radius and ε is very small, we have

$$\varepsilon = \frac{e}{\sin 1''} \sin (\beta + \alpha) = e'' \sin (\beta + \alpha), \tag{134}$$

and the true reading of the circle is

$$R = M + \epsilon'' \sin (\beta + \alpha). \tag{135}$$

The vernier arm MC is usually produced to the opposite point of the circle, which call M', and carries another vernier or microscope. The minutes and seconds of the circle reading at this point being obtained, (a second vernier is not necessary to determine the degrees), if M' is the observed reading, the true reading R is given by

$$R = M' + \epsilon'' \sin (180^\circ + \beta + \alpha) = M' - \epsilon'' \sin (\beta + \alpha).$$

Combining this with (135) we have

$$R = \tfrac{1}{2} (M + M'); \tag{136}$$

from which it appears that the eccentricity is fully eliminated by taking the mean of two readings 180° apart. It can be shown also that it is eliminated by using any number of equidistant microscopes.

THE MICROMETER.

59. Angles which are smaller than the angular diameter of the field of the telescope are most easily and accurately measured by means of a *micrometer*. This is the same as that used in the reading microscope, save that the movable wire is composed of a single thread and is accompanied by other wires. There is generally one *fixed* wire parallel to the movable wire and usually at least one *transverse* fixed wire perpendicular to it. The arrangement of the wires varies to meet the requirements of different problems. The plane of the wires is in the common focus of the object-glass and eye-piece of the telescope and perpendicular to the line of sight. The micrometer is so constructed that it can be revolved about the line of sight until the transverse wire is in, or parallel to, the plane of the angle which it is desired to measure. Then to measure the angle we place the fixed wire on one object, the micrometer wire on the other, and the angle is at once given in terms of a revolution of the micrometer screw, which must be known.

The angle may be measured by means of the movable wire alone. It is first made to bisect one object and then the other,

and the required angle is given by the difference of the two micrometer readings. But in this case a level (§ 61) should be attached to the telescope in such a way as to indicate the amount of any change in its position while the observations are being made.

60. *The angular value of a revolution of the micrometer screw* depends upon the pitch of the screw and the focal length of the telescope. It may be found in several ways.

(a). By the methods described above, measure with the micrometer any known angle, and divide the number of seconds in the angle by the corresponding number of revolutions of the screw.

If the distance between two stars is measured, the true distance must be corrected for refraction.

Example. The difference of declination of *d Ursae Majoris* and *Gr.* 1564 was measured with the micrometer of the zenith telescope of the Detroit observatory, when on the meridian, 1891 March 28. Barom. 29.206 inches, Att. Therm. 58°.0 F., Ext. Therm. 37°.5 F. Find the value of a revolution of the screw. The zenith level was read immediately after bisecting each star, in order to correct for any change in the pointing of the telescope. The value of one division is 2″.74.

Star.	α	Apparent δ	Level.		Microm.
			n	s	
d Ursae Maj.	$9^h\ 25^m$	$+70°\ 18'\ 46''.5$	40.4	18.7	48.566
Gr. 1564.	9 33	$+69\ \ 44\ \ 12\ .7$	41.3	19.5	2.598

The correction for level is $0.85\ d = 2''.3$, by which amount the measured distance must be increased, or the difference of the declinations decreased. The difference of the refractions for the two stars is 0″.7, by which amount the difference of the declinations must be decreased. The corrected difference of the declinations is 34′ 30″.8. Therefore the value of one revolution of the screw is 45″.05.

(b). A more accurate value is obtained by observations on one of the close circumpolar stars. The telescope is directed so that the star is just entering the field, and will be carried through the centre by its diurnal motion. The micrometer is

revolved so that the micrometer wire is parallel to the declination circle through the centre of the field. The wire is set just in advance of the star, the time of transit of the star over it is noted, and the micrometer is read. The wire is moved forward one revolution, or a part of a revolution, and the transit observed as before. In this way the observations are carried entirely across the field.

In Fig. 11, let P be the pole, EP the observer's meridian, ab the diurnal path of a star, AS the position of the micrometer wire when at the centre of the field and coincident with a declination circle PM, and BS' any other position of the wire. Now let m_0 be the micrometer reading, t_0 the hour angle and T_0 the sidereal time when the star is at S, and let m, t and T be the corresponding quantities when the star is at S', and let R be the value of one revolution of the screw. Through S' pass an arc of a great circle $S'C$ perpendicular to AS. Then in the triangle $CS'P$, right-angled at C, we have

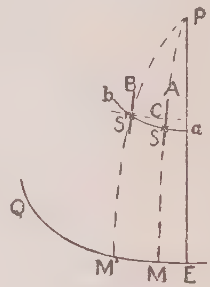

Fig. 11.

$$CS' = (m - m_0)R, \quad S'P = 90^\circ - \delta, \quad CPS' = t - t_0 = T - T_0;$$

and we can write

$$\sin [(m - m_0)R] = \sin (T - T_0) \cos \delta;$$

or, since $(m - m_0)R$ is always small,

$$(m - m_0)R = \sin (T - T_0) \frac{\cos \delta}{\sin 1''}. \tag{137}$$

Similarly, for another observation,

$$(m' - m_0)R = \sin (T' - T_0) \frac{\cos \delta}{\sin 1''}.$$

Combining these to eliminate the zero point,

$$(m' - m)R = \sin (T' - T_0) \frac{\cos \delta}{\sin 1''} - \sin (T - T_0) \frac{\cos \delta}{\sin 1''}, \tag{138}$$

from which the value of R is obtained. The micrometer readings are supposed to increase with the time.

The times of transit are supposed to be noted by means of a sidereal time-piece. If its rate [§ 66] is large it must be allowed

for. If a mean time-piece is used the intervals $T - T_0$ must be converted into sidereal intervals.

The resulting value of R is slightly in error on account of refraction, since the star is observed at unequal zenith distances. But the effect of refraction is inappreciable if the observations are made near the meridian. The method is therefore advantageous for a meridian instrument with a micrometer in right ascension, the star being observed at upper or lower culmination. However, any variations in the azimuth or level constants of the instrument during the progress of the observations introduce errors in the results. If a and b are the values of these constants at the beginning of the series of transits and a' and b' their values at the close of the series, we can show later, from the theory of the transit instrument, that the distance between the first and last positions of the wires has been decreased by the quantity

$$(a' - a) \sin(\phi \mp \delta) + (b' - b) \cos(\phi \mp \delta), \tag{139}$$

which divided by the corresponding difference of the micrometer readings is the correction to the value of one revolution of the screw. The lower signs are for lower culmination.

The azimuth constants are determined by observing suitable pairs of stars before and after the series of micrometer transits is taken, according to the methods described later. The level constants are determined by the method of § 61. The variations of azimuth and level may be considered to be uniform and proportional to the time. For a meridian instrument properly mounted the variation of the azimuth may be neglected without a sacrifice of accuracy.

Example. *Polaris* was observed at lower culmination at Ann Arbor, 1891 March 28, to determine the value of a revolution of the micrometer screw of the transit instrument. The micrometer was set at every three-tenths of a division, and one hundred and fifty transits observed. The times were noted by means of a sidereal chronometer which was $16^m\ 30^s.6$ slow. The position of *Polaris* was

$$\alpha = 1^h\ 17^m\ 46^s.0, \qquad \delta = +88^\circ\ 43'\ 40''.25,$$

Nautical Almanac, p. 304; and therefore the chronometer time of lower culmination was $T_0 = 13^h\ 1^m\ 15^s.4$. A few of the observations and their reduction are given below. Each

observation is the mean of three consecutive original observations.

m	T	$T - T_0$	$T - T_0$	$\sin(T - T_0)$	$(m - m_0)R$
7.8	$12^h\ 23^m\ 12^s.3$	$-38^m\ 3^s.1$	$-9°\ 30'\ 46''.5$	9.218194$_n$	$-756''.83$
8.7	25 17 .3	35 58 .1	8 59 31 .5	9.193953$_n$	715 .75
9.6	27 19 .7	33 55 .7	8 28 55 .5	9.168793$_n$	675 .46
10.5	29 22 .0	31 53 .4	7 58 21 .0	9.142070$_n$	635 .15
11.4	31 22 .0	29 53 .4	7 28 21 .0	9.114111$_n$	595 .55
20.4	51 42 .3	9 33 .1	2 23 16 .5	8.619771$_n$	190 .80
21.3	53 43 .3	7 32 .1	1 53 1 .5	8.516822$_n$	150 .53
22.2	55 46 .0	5 29 .4	1 22 21 .0	8.379348$_n$	109 .69
23.1	57 47 .0	3 28 .4	0 52 6 .0	8.180547$_n$	69 .40
24.0	59 50 .0	$-$ 1 25 .4	$-$0 21 21 .0	7.793121$_n$	$-$ 28 .44
25.8	13 3 54 .0	+ 2 38 .6	+0 39 39 .0	8.061960	+ 52 .82
26.7	5 58 .3	4 42 .9	1 10 43 .5	8.313268	94 .21
27.6	7 59 .5	6 44 .1	1 41 1 .5	8.468092	134 .55
28.5	10 0 .0	8 44 .6	2 11 9 .0	8.581389	174 .66
29.4	12 3 .7	10 48 .3	4 42 4 .5	8.673281	215 .82
38.4	32 24 .3	31 8 .9	7 47 13 .5	9.131914	620 .48
39.3	34 28 .3	33 12 .9	8 18 13 .5	9.159630	661 .20
40.2	36 32 .2	35 16 .8	8 49 12 .0	9.185629	702 .16
41.1	38 34 .0	37 18 .6	9 19 39 .0	9.209722	742 .21
42.0	13 40 37 .0	+39 21 .6	+9 50 24 .0	9.232735	+782 .60

Substracting the 1st from the 11th, the 2nd from the 12th, etc., we have

$m' - m$	$(m' - m)R$	R	v	v^2
18.0	$809''.65$	$44''.981$	$-0''.065$	0.0042
18.0	809 .96	44 .998	$-$0 .048	0.0023
18.0	810 .01	45 .001	$-$0 .045	0.0020
18.0	809 .81	44 .989	$-$0 .057	0.0032
18.0	811 .37	45 .076	+0 .030	0.0009
18.0	811 .28	45 .071	+0 .025	0.0006
18.0	811 .73	45 .096	+0 .050	0.0025
18.0	811 .85	45 .103	+0 .057	0.0032
18.0	811 .61	45 .089	+0 .043	0.0018
18.0	811 .04	45 .058	+0 .012	0.0001
		$R = 45$ .046		$\sum v^2 = 0.0208$

$$\text{Probable error}^* = \pm 0.674 \sqrt{\frac{0.0208}{10 \times 9}} = \pm 0''.010.$$

By reducing the whole series of transits the value of R and its probable error were found to be

$$R = 45''.059 \pm 0''.006.$$

* See any work on the *Method of Least Squares.*

From the level readings $b = +\ 5''.17$, $b' = +\ 7''.05$, and from observations for azimuth on β *Cassiopeae* and 4 *H. Draconis,* and on θ *Bootis* and 36 *H. Cassiopeae,* $a = -\ 9''.15$, $a' = \quad 8''.79$. Substituting these in (139) and dividing by 46, the difference of the first and last micrometer readings, we have as a correction to R, $-\ 0''.017$, and therefore

$$R = 45''.042 \pm 0''.006.$$

It is clearly evident from the individual results for R that its value increases as the micrometer readings increase. This irregularity should be fully investigated by further observations, and allowed for in refined observations.

The value of a revolution is affected by changes of temperature. To determine the rate of change, observations should be made several nights at widely different temperatures. If R is the value of a revolution at the temperature τ, R_0 the value at the temperature 50°, and x the correction to R_0 for a rise of 1° in temperature, each night's observations furnish an equation of the form

$$R = R_0 + (\tau - 50^\circ)x. \tag{140}$$

The solution of these equations by the method of least squares gives the most probable value of R_0 and x, and therefore of R.

(c). If the micrometer is designed for the measurement of zenith distances, the micrometer wire being horizontal, the observations are made at the time of the star's *greatest western* or *eastern* elongation. This occurs when the vertical circle of the star is tangent to its diurnal circle. At this time the micrometer wire is parallel to the star's declination circle. If m_0, t_0 and T_0 refer to the instant of greatest elongation, and m, t and T to any other instant, the formula (137) is applicable to this case. At the instant of greatest elongation the parallactic angle ZOP, Fig. 1, is 90° for western and 270° for eastern elongation, and we can write

$$\cos t_0 = \tan\phi \cot\delta, \quad \cos z_0 = \sin\phi \operatorname{cosec}\delta, \quad T_0 = a + t_0. \tag{141}$$

Set the telescope at the zenith distance z_0 when the star is just entering the instrument. Note the time of transit over the micrometer wire; and, as before, carry the observations across the field. Any change in the zenith distance of the telescope during the progress of the observations will affect the resulting

value of R. The amount of the change will be indicated by the zenith level and can be allowed for. The level should be read after each transit is observed. If l_0 is the level reading at the time T_0, l the level reading at the time T, and d the value in arc of a division of the level, we have

$$(m - m_0) R = \pm \sin (T - T_0) \frac{\cos \delta}{\sin 1''} + (l - l_0) d; \tag{142}$$

and for another observation

$$(m' - m_0) R = \pm \sin (T' - T_0) \frac{\cos \delta}{\sin 1''} + (l' - l_0) d.$$

Whence

$$(m' - m) R = \pm \sin (T' - T_0) \frac{\cos \delta}{\sin 1''} \mp \sin (T - T_0) \frac{\cos \delta}{\sin 1''} + (l' - l) d, \tag{143}$$

in which the lower sign is for eastern elongation. The micrometer readings are supposed to increase with the time for western elongations, and the level readings to increase toward the north.

The resulting value of R must be corrected for refraction. From the values of z_0 and R the zenith distances corresponding to the first and last observations can be obtained, and thence the refractions. The difference of the refractions divided by the difference of the first and last micrometer readings is the amount by which the value of R must be decreased.

If both R and d are unknown a close approximation to the value of R is obtained by neglecting the term $(l' - l)d$. With this value of R the value of d is computed (§ 64) and substituted in (143), and the corrected value of R obtained. A second approximation to the value of d will rarely be required.

THE LEVEL.

61. The *spirit level* consists of a sealed glass tube, ground on the upper interior surface to the arc of a circle of large radius, and nearly filled with alcohol or ether. The bubble of air occupying the space not filled by the liquid is always at the highest point of the curve. Therefore a change in the relative elevations of the ends of the tube causes a motion of the bubble, the amount of which is read from a scale marked on the surface of the glass. The level is adapted to the determination of the angle which a nearly horizontal line makes with the horizon, or the very small angle moved over by a telescope.

The level tube is mounted and attached to astronomical instruments in various ways, but there is one general method of using it. Let the divisions of the scale be numbered in both directions from zero at the centre, and d be the angular value of one division. If the level be placed on a truly horizontal line—say, for convenience, on east and west line—the centre of the bubble will not be at zero, owing to the non-adjustment of the level. If the centre is x divisions from the zero, the error of the level is dx. Now let the level be placed on a line inclined to the horizon at an angle b, and let the reading of the west end of the bubble be w and the east end e. Then the elevation of the west end of the line is given by

$$b = \tfrac{1}{2}(w - e)d \mp dx.$$

Now let the level be reversed in direction and let the reading of the west end be w' and the east end e'. Then

$$b = \tfrac{1}{2}(w' - e')d \pm dx.$$

Combining these values of b we have

$$b = \tfrac{1}{4}[(w + w') - (e + e')]d; \tag{144}$$

from which it appears that the error of the level is eliminated by reversing.

Whenever it is possible the level should be read several times, the same number of readings being made in each position,—level *direct* and level *reversed*,—care being taken to remove the level from its bearings after each reading is made.

Example. The inclination of the axis of a transit instrument is required from the following level readings, the value of one division being $1''.88$.

	w	e	
Direct	14.1	9.7	53.4
Reversed	12.6	11.1	41.8
Reversed	12.7	11.1	8)11.6
Direct	14.0	9.9	1.45
Sum	53.4	41.8	

The axis makes an angle $1.45\,d = 2''.73$ with the horizon, the west end being higher than the east.

62. In case the zero of the scale is at one end of the tube and the numbers increase continuously to the other, we can show that

$$b = \tfrac{1}{4}[(w + e) - (w' + e')]. \tag{145}$$

If w is greater than e, and therefore w' less than e', b is the elevation of the west end of the line.

Example. Find the inclination of the axis of a transit instrument from the following level readings, the value of one division being $2''.743$.

	Direct.		Reversed.	
w	35.4	w'	16.3	95.3
e	12.4	e'	39.4	111.6
w	35.3	w'	16.4	8) — 16.3
e	12.2	e'	39.5	— 2.037
Sum	95.3	Sum	111.6	

The axis makes an angle $-2.037\,d = -5''.59$ with the horizon, the west end being lower than the east.

63. The *value of one division* of the level is determined best by means of a *level-trier*. This consists of a horizontal bar supported at one end by two bearings and at the other by a vertical micrometer screw. The level is placed on the bar and the readings of the micrometer and bubble are noted. The screw is now turned and the bubble moves to a new position. The readings of the micrometer and bubble are again noted. The angle moved over by the bar is known from the length of the bar, the pitch of the screw and the difference of the micrometer readings; whence the angular value of one division of the level may be obtained.

64. Another convenient method is to attach the level to a telescope provided with a micrometer in zenith distance, the level being placed so that the vertical plane passing through it is parallel to the line of sight. The telescope is directed to a distant terrestrial mark and the level set so that the bubble is at one end of the scale. The mark is bisected by the micrometer wire and the level and micrometer readings noted. The instrument is then turned through an angle such that the bubble moves to the other end of the scale. The mark is again bisected by the wire and the level and micrometer readings noted as before. The difference of the level readings corresponds to the difference of the micrometer readings, whence the value of one division of the level can be obtained from the known value of a revolution of the micrometer screw.

Example. The following observations were made February 19, 1891, to determine the value of a division of the striding level of the Detroit Observatory transit instrument, the teles-

cope being directed to a distant mark. The value of one revolution of the screw is 45″.042. Find the value of a division of the level.

Level.		Micrometer.	Differences.		d
n	s		Level.	Microm.	
20.9	1.1	17.019			
2.0	20.0	17.791	18.9	0.772	0.0408 R
1.8	20.2	17.773			
20.6	1.4	16.969	18.8	0.804	0.0428 R

The mean of eighteen observations gave $d = 0.0417\ R \pm 0.0007\ R = 1''.878 \pm 0''.009$.

65. The level tube should be thoroughly tested for irregularity of curvature before using. If different portions of a level give sensibly different values for a division of the scale, it should not be used in refined observations.

The value of a division should also be determined at two or more very different temperatures in order that a temperature correction may be introduced if necessary.

A level should be adjusted by the vertical adjusting screws so that the bubble will stand near the center of the tube when the level is placed on a horizontal line. It should be adjusted by the horizontal screws so that the axis of the tube will be parallel to the line whose inclination is to be measured. This adjustment is tested by revolving the level slightly about its bearings. If the reading changes the adjustment is not perfect.

THE CHRONOMETER.

66. A *chronometer* is a large and carefully constructed watch which is "compensated" so that changes of temperature have very little effect on the time in which the balance-wheel vibrates. It is a very accurate time-piece when properly handled, comparing favorably with the astronomical clock, and being portable is adapted to field work and navigation.

The *chronometer correction* is the amount which must be added to the reading of the chronometer-face to obtain the correct time. It is + when the chronometer is slow. The *chronometer*

rate is the daily increase of the chronometer correction. It is + when the chronometer is losing. It is not necessary that the correction and rate be small, though it is convenient to have the rate less than $\pm 5^s$ a day. The test of a good time-piece lies in the uniformity of its rate. The correction is generally allowed to increase indefinitely.

The chronometer correction is obtained by observations on the celestial objects, or by comparison with a time-piece whose correction is known. If

$\Delta T_0 =$ the chronometer correction at a time T_0,
$\Delta T =$ the chronometer correction at a time T,
$\delta T =$ the chronometer rate,

we determine the rate per unit of time by

$$\delta T = \frac{\Delta T - \Delta T_0}{T - T_0}. \tag{146}$$

Conversely, if the rate and the correction at the instant T_0 are known, the correction at the instant T is given by

$$\Delta T = \Delta T_0 + \delta T\,(T - T_0). \tag{147}$$

Example. The correction to chronometer T. S. & J. D. Negus, no. 721, was $+ 16^m\ 19^s.5$ at Ann Arbor mean time 1891 March $25^d\ 11^h$, and $+ 16^m\ 55^s.6$ at 1891 April $4^d\ 11^h$. Find the daily rate and the correction at 1891 March $28^d\ 13^h$.

From (146) we find the daily rate $\delta T = + 3^s.61$. Subtituting this and $T =$ March $28^d\ 13^h$ in (147) we find

$$\Delta T = + 16^m\ 19^s.5 + 3^s.61 \times 3.08 = + 16^m\ 30^s.6.$$

The above equations are true only when the rate is constant for the interval $T - T_0$. Such constancy can be assumed for an interval of a few days in the case of the best chronometers; but when great accuracy is required the interval between observations for determining chronometer correction should be as small as possible.

It is convenient to use a sidereal chronometer when making observations on the stars, and a mean time chronometer when making observations on the sun.

67. The observer should be able to "carry the beat" of the chronometer; that is, to mentally count the successive seconds from the tick of the chronometer without looking at it. An

experienced observer will carry the beat for several minutes, estimate the times of transits of the star over several wires to the tenths of seconds, and write them on a slip of paper without taking his eye from the telescope: then, still carrying the beat, he will look at the chronometer face to verify his count. This is called the "eye and ear method" of observing.

To obtain the best results from a chronometer the following precepts should be rigidly observed:

(*a*). It should be wound at regular intervals. If it requires winding daily it should always be wound at the same hour of the day; otherwise an unused part of the spring is brought into action and a change of rate results.

(*b*). The hands should not be moved forward oftener than is necessary, and they should never be moved backward.

(*c*). A chronometer on ship board should be allowed to swing freely in its gimbals, so that it may always take a horizontal position; but when carried about on land it should be clamped so as to avoid the violent oscillations due to the sudden motions it receives.

(*d*). It should be kept in a dry place; as nearly at a uniform temperature as possible; away from magnetic influences; and when at rest should always be in the same position with respect to the points of the compass.

(*e*). All quick motions should be avoided: in particular, it should never be rotated rapidly about its vertical axis.

68. When several chronometers are employed, the correction to one is obtained by observation; and to the others, by comparison with the first. If two chronometers which keep the same kind of time are compared it will generally happen that they do not beat together. The fraction of a second by which one beats later than the other can be estimated after some practice to within $0^s.1$ or $0^s.2$, so that the correction can be obtained to that degree of accuracy by this method.

When a sidereal chronometer is compared with a mean time chronometer the degree of accuracy is higher. If the chronometers tick half seconds the beats of the two will coincide once in every 183^s, or very nearly in 3^m, since in this interval sidereal time gains $0^s.5$ on mean solar. The ear is capable of estimating the coincidence of the beats within $0^s.02$ or $0^s.03$. When the coincidence occurs the observer notes the times indicated by the two chronometers. The correction to the one being

known the very approximate correction to the other is readily obtained.

When a chronograph [§ 70] is at hand and the chronometers are provided with break-circuits (or make-circuits), the comparisons are most conveniently and accurately made by placing the two chronometers in the chronograph circuit. The beats are recorded on the chronograph sheet and the distance between them can be measured very accurately by means of a scale.

69. The *astronomical clock* is a finely constructed clock whose pendulum is compensated for changes of temperature. Its rate is in general more uniform than that of a chronometer. It is one of the *fixed* instruments of an observatory, and to that extent the remarks concerning the chronometer are applicable to it.

THE CHRONOGRAPH.

70. The *chronograph* is a mechanical device for recording the instant when an observation is made. A sheet of paper on which the record is to be made is wrapped around a metallic cylinder which is caused to rotate once per minute by means of clock-work. A pen is attached to the armature of an electro-magnet in such a way as to press its point on the moving paper. The magnet is carried slowly along the cylinder by a screw, so that the pen traces a continuous spiral on the paper. The electro-magnet is placed in an electric circuit which passes through the chronometer or clock in such a way that the circuit is broken for an instant at the beginning of every second, or every other second. At each of these instants the electro-magnet releases the armature carrying the pen, the pen moves laterally for the moment, and in this way the spiral is graduated by notches to seconds of time. The notch is usually omitted at the beginning of the minute. One of the circuit wires passes through a signal-key held in the observer's hand. When a star, for example, is being observed he presses the key at the exact instant when the star is crossing a wire, thus breaking the circuit and making the record on the chronograph sheet. The beats of the chronometer being recorded on the sheet, the chronometer time when the key was pressed can be read from the sheet by means of a scale with great accuracy and at the observer's leisure.

When the chronograph is set in motion the observer notes in some way the hour, minute and second corresponding to a

certain point on the sheet, which serves as a reference point in reading the sheet.

In some forms of the chronograph the circuit is *made* by pressing the key, but the *break-circuit* is preferable.

The chronographic method is preferable to the eye and ear method because it relieves the mind from carrying the beat and making the record, thus allowing greater care to be given to other parts of the observation; and because more observations can be made in a given time and the instant of transit can be read off to the hundredth of a second.

CHAPTER VII.

THE SEXTANT.

71. The sextant is an instrument especially adapted to the determination of time, latitude and longitude when extreme accuracy is not required, as in navigation and exploration. It consists essentially of a brass frame ADC, Fig. 12, bearing a graduated arc AC, a telescope EF, whose line of sight is parallel to the plane of the graduated arc, and the mirrors H and D, whose planes are perpendicular to the plane of the arc. The mirror D, called the *index-glass*, is fixed to the *index-arm* DB, which revolves about D at the centre of the arc, and which carries a vernier at B. The mirror H, called the *horizon-glass*, is attached to the frame. The lower half of it is silvered, the upper half is left clear.

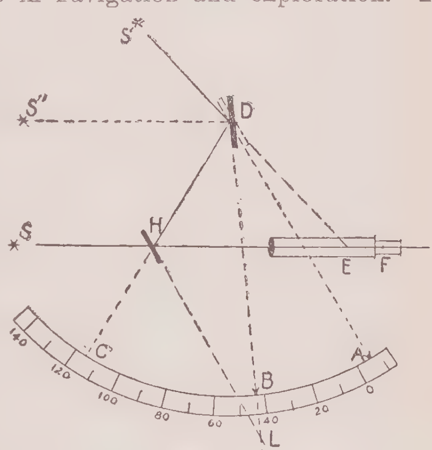

Fig. 12.

72. To illustrate the method of using a sextant and the principles involved, let it be required to measure the angle SES' between the stars S and S'. The instrument is held in the hand and the telescope directed to the star S. The ray SE passes through the unsilvered part of H and forms a *direct* image of the star at the focus F. The sextant is revolved about

the line of sight until its plane passes through the other star S'. The index-arm is then moved until the *reflected* image of S' is brought into the field and nearly in coincidence with the direct image of S. The index-arm is clamped and the two images brought into perfect coincidence by turning the slow-motion or tangent screw. If the instrument is perfectly constructed and adjusted, the required angle is given at once by the circle reading. The ray of light $S'D$, which forms the reflected image at F, traverses the path $S'D$-DH-HF, being reflected by the two mirrors D and H. When the direct and reflected images coincide the angle between the stars is twice the angle between the mirrors. That is, $SES' = 2\,HLD$. For the angle

$$\begin{aligned} SES' &= 180^\circ - EDH - EHD \\ &= 180^\circ - 2HDL - 2(LHD - 90^\circ) \\ &= 2(180^\circ - HDL - LHD) \\ &= 2HLD. \end{aligned}$$

If A is the position of the zero of the vernier when the two mirrors are parallel, and B its position when the two images coincide, we have

$$SES' = 2HLD = 2ADB = 2AB. \tag{148}$$

It thus appears that to enable us to read the required angle directly from the circle, the circle reading must be twice the corresponding arc. Thus, the 120° line is really only 60° from the 0° line [or a sextant, hence the name].

In the prismatic sextant of Pistor & Martins the horizon-glass is replaced by a prism, which has the advantage that any angle up to 180°, or even greater, can be measured; whereas with the common sextant the angle is limited to about 140°.

73. In order to obtain good results the instrument is accurately adjusted, and the telescope focused; the images are made equally bright and are brought into coincidence in the centre of the field of view; and several observations are made in quick succession, the mean or all being used. The coincidence of the images is tested by vibrating the instrument slightly about the line of sight, thus causing the reflected image to pass by or over the direct one. The images are made equally bright by moving the telescope from or toward the frame by a screw, so as to utilize more or less of the light passing through the transparent part of the horizon-glass, or by placing colored-glass shades in front of the index-glass, as in observations on the sun or moon.

In measuring the distance of the moon from a star, the star is brought into coincidence with that point of the moon's bright limb which lies in the great circle joining the star and the centre of the moon. The measured distance is then increased by the moon's semidiameter [§§ 33, 34, 35]. In the case of the sun and moon the images of the nearest limbs are made to coincide, and the measured distance is increased by the semidiameters as before. Results obtained in this way when corrected for any instrumental errors are the apparent distances between the objects.

74. The sextant is used also for measuring the apparent altitudes of the heavenly bodies. At sea the telescope is directed to that point of the horizon which is below the object. The reflected image is brought into contact with the horizon line. When the instrument is vibrated the image should describe a curve tangent to the horizon. The sextant reading corrected for instrumental errors and the dip of the horizon [§ 32] is the apparent altitude. If the object is the sun, the lower or upper limb is made tangent to the horizon; if the moon, the bright limb; and the sextant readings must be further corrected for semidiameter.

75. For observing altitudes on land an artificial horizon is used. This is a shallow rectangular basin of mercury over which is placed a roof, made of two plates of glass set at right angles to each other in a frame, to protect the mercury from agitation by the wind. The mercury forms a very perfect horizontal mirror which reflects the rays of light from the star. If the observer places his eye at some point in a reflected ray, he will see an image of the star in the mercury, whose angle of of depression below the horizon is equal to the altitude of the star above the horizon. If then he directs the telescope to the image in the mercury, and brings the two images into coincidence as before, the sextant reading corrected for instrumental errors is double the apparent altitude of the star. The sun's altitude is measured by making the two images tangent externally. The corrected sextant reading is double the altitude of the lower or upper limb, according as the nearest or farthest limbs of the sun and its image in the mercury are observed.

The double altitudes of stars near the meridian are changing slowly, and the images are brought into contact by means of the slow-motion screw as before. But the double altitudes of stars at a distance from the meridian are changing rapidly and another

method is used. To illustrate, suppose the sun is observed for time when it is east of the meridian, and the altitude therefore increasing. The upper limb is observed first. The two images are brought into the field and the index moved forward until the sextant reading is from 10′ to 20′ greater than the double altitude of the upper limb, and the instrument is clamped. The images are now slightly separated, but they are approaching. When they become tangent, the observer notes the time on the chronometer. The index is again moved forward from 10′ to 20′ and the contact observed as be fore. In this way four or five observations are made. The double diameter of the sun is is abont 64′, and for observing the lower limb the index is quickly moved backward about 45′. The two images now overlap, but they are separating, and the time is noted when they become tangent. Moving the index forward as before, four or five observations are made on the lower limb.

If the sun is observed west of the meridian, the altitudes of the lower limb should be measured first.

76. The faces of the glass in the horizon roof should be perfectly parallel. If they are prismatic the observed altitudes are erroneous. The error is eliminated by observing one-half of a set of altitudes with the roof in one position and the other half with the roof in the reversed position, and taking the mean of all.

The surface of the mercury can be freed from impurities by adding a little tin-foil. The amalgam which forms can be drawn to one side of the basin by means of a card, leaving a perfectly bright surface.

ADJUSTMENTS OF THE SEXTANT.

77. (a). *The index-glass.* Place the sextant on a table, unscrew the telescope and set it in a vertical position on the graduated arc. Place the eye near the index-glass and move the index-arm toward the telescope until the telescope and its image in the mirror are seen very nearly in coincidence. Their corresponding outlines will be parallel if the index-glass is perpendicular to the plane of the arc. If they are not parallel the glass is removed and one of the points against which it rests is filed down the proper amount. The axis of the telescope is here assumed to be perpendicular to the plane of the end on which it rests. This can be tested by rotating the telescope about its axis and noticing whether the angle between the tube

and its image varies. The telescope should be set at the mean of the two positions which give the maximum and minimum values of this angle.*

(*b*). *The horizon-glass.* The index-glass having been adjusted, the telescope is directed to a star and the index-arm is brought near the zero of the arc. If the horizon-glass is parallel to the index-glass the reflected image will pass through the direct image when the index-arm is moved slowly to and fro. If it passes on either side of the direct image the horizon-glass needs adjustment. This is done by turning the screws provided for the purpose.

(*c*). *The telescope.* Two parallel wires are placed in the telescope tube. These are made parallel to the plane of the sextant by revolving the tube containing them. The line of sight is the line joining the points midway between these wires and the centre of the object glass. This should be parallel to the plane of the sextant. To test the adjustment, select two well defined objects about 120° apart, and bring the two images into coincidence on one of the side wires, and then move the sextant so as to bring the images on the other wire. If the images still coincide, the line of sight needs no adjustment. If the images are separated, the collar which holds the telescope is shifted by means of screws until the adjustment is satisfactory.

CORRECTIONS TO SEXTANT READINGS.

78. *The index correction.* It is seen from (148) that all angles measured by the sextant are reckoned from A, the point where the zero of the vernier falls when the two mirrors are parallel; whereas the circle readings are measured from 0°. The *index correction* is the reading 0° (or 360°) minus the reading at A. Let it be represented by I. The value of I can be reduced to zero by rotating slightly the horizon-glass by means of screws provided for that purpose. But this adjustment is very liable to derangement, and it is customary to determine I every time the sextant is used and apply it to all the sextant readings.

(*a*). To determine I for correcting stellar observations, point the telescope to a star and bring the direct and reflected

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* This method was proposed by Professor J. M. Schaeberle. See *The Sidereal Messenger* for May, 1888.

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images into coincidence. Let the sextant reading be R. The index correction is given by

$$I = 0^\circ - R. \tag{149}$$

Example. Determine I from the following readings:

359° 56′ 0″	359° 55′ 50″
56 10	56 10
55 50	55 55

The mean of the six readings is 359° 55′ 59″.2, and therefore

$$I = 360^\circ - 359^\circ\ 55'\ 59''.2 = +\ 4'\ 0''.8.$$

(b). For reducing solar observations, point the telescope to the sun and bring the direct and reflected images externally tangent to each other and read the circle. Then move the reflected image over the direct image until they are again externally tangent, and read the circle. Let the readings in the two positions be R_1 and R_2, R_1 being the greater. The reading when the two images coincide is $\frac{1}{2}(R_1 + R_2)$, and the index correction is given by

$$I = 360^\circ - \tfrac{1}{2}(R_1 + R_2). \tag{150}$$

The observed semidiameter of the sun is given by

$$S = \tfrac{1}{4}(R_1 - R_2). \tag{151}$$

To eliminate the effect of refraction the horizontal semidiameter should be measured.

Example. Find I and S from the following readings on the sun made Thursday, April 23, 1891.

	360° 28′ 35″	359° 24′ 50″
	40	40
	45	45
	45	45
	40	50
Means	360 28 41 .0	359 24 46 .0

$$I = 360^\circ - 359^\circ\ 56'\ 43''.5 = +\ 3'\ 16''.5.$$
$$S = \tfrac{1}{4}(1\ \ 3'\ 55''.0) = 15'\ 58''.7.$$

From the Nautical Almanac, p. 56, $S = 15'\ 56''.3$.

79. *Correction for eccentricity.* The arc of a sextant being short, the eccentricity cannot be eliminated by means of two verniers 180° apart, and it must be investigated. This can be

done by comparing several angles measured with the sextant with their known values obtained in some other way. Thus in Fig. 10 the sextant reading is twice the arc AM. The true value of the angle is obtained by correcting the reading at M for eccentricity, and correcting the position of A for eccentricity and index correction. The true reading at M is given by (135). The true reading at the zero point A is given by

$$R_0 = -I + e'' \sin \alpha.$$

The true value of the angle is

$$R - R_0 = M + e'' \sin(\beta + \alpha) - e'' \sin \alpha + I. \tag{152}$$

But $R - R_0$ is the known value of the angle; let d represent it. M is the observed value of the angle; let d' represent it. Now, since an arc on the sextant is one-half the corresponding reading, we have

$$\tfrac{1}{2}(d - d') = e'' \sin(\beta + \alpha) - e'' \sin \alpha + \tfrac{1}{2} I,$$

which reduces to

$$\begin{aligned} d - d' &= 4\,e'' \cos(\tfrac{1}{2}\beta + \alpha) \sin \tfrac{1}{2}\beta + I \\ &= 4\,e'' \cos \alpha \sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}\beta - 4\,e'' \sin \alpha \sin^2 \tfrac{1}{2}\beta + I. \end{aligned} \tag{153}$$

If we put

$$4\,e'' \cos \alpha = x, \qquad 4\,e'' \sin \alpha = y, \tag{154}$$

we have

$$\sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}\beta\, x - \sin^2 \tfrac{1}{2}\beta\, y + I = d - d'. \tag{155}$$

This equation involves three unknown quantities, x, y and I. Three measured angles, each furnishing an equation of the form (155), are required for the solution of the problem.

There are several ways in which to obtain the value of $d - d'$ at any point on the arc.

(a). For those who have access to a meridian circle, the most direct process is the ingenious method proposed by Professor Schaeberle in *Astronomische Nachrichten*, no. 2832.

(b). When the latitude of the observer and the time are accurately known, make a series of measures of the double altitudes of a northern star just before and after its meridian passage. The observed double altitude at the instant of transit is obtained from these measures by the method of § 91. The

apparent double altitude at the instant is obtained at once from the known declination, latitude and refraction. The latter minus the former is $d - d'$.

(c). When the latitude and time are not accurately known, measure the distance between two stars and compare it with the known apparent distance. The apparent distance is found by § 20, using a', δ' and a'', δ'' as affected by refraction, § 31.

Example. The distance between *Aldebaran* and *Arcturus* was measured with the sextant at Ann Arbor, 1891 March 5, as below. It is required to form the equation (155) for this pair of stars. The chronometer correction $\Delta\theta$ was $+ 15^m\ 7^s$.

	Chronometer.	Sextant.
	$8^h\ 37^m\ 25^s$	130° 14′ 55″
	45 30	14 40
	48 30	14 55
	52 20	14 35
	56 20	14 55
	9 0 30	14 45
	9 4 0	14 40
Means	8 52 5	130 14 46.4
$\Delta\theta$	+ 15 7	
θ	9 7 12	

Barom. 29 .400 inches.
Att. Therm. 65°.0 F.
Ext. Therm. 18°.0 F.

From the Naut. Alm. pp. 322, 340,

	Aldebaran.	*Arcturus.*
a	$4^h\ 29^m\ 39^s$ 33	$14^h\ 10^m\ 41^s$ .97
δ	16° 17′ 22″.7	19° 44′ 47″.8

With this data we solve (40), (35), (36), (37), (32), (80), (84) and (85) as below.

	Aldebaran.	*Arcturus.*
θ	$9^h\ 7^m 12^s$	$9^h\ 7^m 12^s$
a	4 29 39	14 10 42
t	4 37 33	18 56 30
t	69°23′ 15″	284° 7′ 30″
ϕ	42 16 47	42 16 47
tan ϕ	9.95870	9.95870
cos t	9.54660	9.38746
L	68°50′ 6″	74°58′ 36″
δ	16 17 23	19 44 48
tan t	0.42467	0.59921_n
cos L	9.55758	9.41366
cosec$(L-\delta)$	0.10027	0.08542
tan q	0.08252	0.09829_n
q	50°24′ 39″	308°34′ 16″
tan $(L-\delta)$	0.11573	0.15849
cos q	9.80433	9.79482
z	63°58′ 41″	66°35′ 42″

sin t	9.97127	9.98667_n
cos ϕ	9.86915	9.86915
cosec z	0.04642	0.03729
cosec q	0.11316	0.10689_n
log 1	0.00000	0.00000
True z	63° 58′ 41″	66° 35′ 42″
Mean refr.	1 56	2 11
App. z	63 56 45	66° 33 31
log μ	1.75821	1.75766
tan z	0.31078	0.36292
A log BT	9.99583	9.99583
λ log γ	0.02746	0.02750
log r	2.09228	2.14391
sin q	9.88684	9.89311_n
sec δ	0.01780	0.02632
log da	1.99692	2.06334_n
cos q	9.80433	9.79482
$d\delta$	+ 1′ 18″.8	+ 1′ 26″.8
da	+ 1 39 .3	− 1 55 .7

Applying these refractions to the above star places we obtain the co-ordinates which are to be used in solving (46), (47), (48) and (43).

α'	$67^\circ\ 26'\ 29''.2$	$\cot(\delta''+G)$	9.914333_n
δ'	$16\ \ 18\ \ 41\ .5$	$\cos B'$	9.842600
α''	$212\ \ 38\ \ 33\ .8$	d	$130^\circ\ 17'\ 22''.4$
δ''	$19\ \ 46\ \ 14\ .6$		
$\alpha''-\alpha'$	$145\ \ 12\ \ \ 4\ .6$	$\sin(\alpha''-\alpha')$	9.756404
$\cot\delta'$	0.533668	$\cos\delta'$	9.982158
$\cos(\alpha''-\alpha')$	9.914429_n	$\operatorname{cosec} B'$	0.143841
G	$289^\circ\ 36'\ 52''.7$	$\operatorname{cosec} d$	0.117597
$\sin G$	9.974038_n	$\log 1$	0.000000
$\tan(\alpha''-\alpha')$	9.841975_n		
$\sec(\delta''+G)$	0.197546	d'	$130^\circ\ 14'\ 46''.4$
$\tan B'$	0.013559	$d-d'$	$+2\ \ 36\ .0$
B'	$45^\circ\ 53'\ 39''.3$	$d-d'$	$+\ \ 156\ .0$

The angle β in (155) is not $\frac{1}{2}d'$, but one-half the reading corresponding to the line of the circle with which the vernier line coincides, and it is the eccentricity of this point which enters into $d-d'$. For the reading $d'=130^\circ\ 14'\ 50''$ the 29th line of the vernier coincides with the circle line $135^\circ\ 0'$, and therefore in this case $\frac{1}{2}\beta = 33^\circ\ 45'$. We now find

$$\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\beta = 0.462, \qquad \sin^2\tfrac{1}{2}\beta = 0.309;$$

and therefore

$$0.462x - 0.309y + I = 156.0.$$

Similarly, from the meridian double altitude of a star, method (*b*), and from another pair of stars we find

$$0.259x - 0.072y + I = 165.0,$$
$$0.117x - 0.014y + I = 171.0.$$

Solving these three equations we obtain

$$\log x = 1.61380_n, \qquad \log y = 0.43265, \qquad I = 175.8;$$

whence, from (154),

$$a = 176^\circ\ 14', \qquad 4e'' = 41''.2.$$

While the index correction varies from day to day and its value should be determined by the methods of § 78 every time the sextant is used, the eccentricity is practically constant. By

neglecting the term I in (153) and making 2β successively $0°$, $10°$, etc., we obtain the following corrections for eccentricity to be applied to the circle readings.

Circle.	Correction.	Circle.	Correction.	Circle.	Correction.
0°	0″.0	50°	— 8″.8	100°	— 16″.2
10	— 1 .8	60	— 10 .5	110	— 17 .4
20	— 3 .6	70	— 12 .0	120	— 18 .5
30	— 5 .4	80	— 13 .5	130	— 19 .4
40	— 7 .2	90	— 14 .9	140	— 20 .2

In order to determine the eccentricity very accurately, at least ten known angles distributed uniformly from 0° to 140° should be measured and the resulting equations solved by the method of least squares. The observations should be made in one night so that I may be considered constant.

DETERMINATION OF TIME.

80. *Time* is *determined* from observations on the heavenly bodies by determining the corrections to the chronometer or other time-piece at the instants when the observations are made.

81. *By equal altitudes of a fixed star.* When a star is from two to four hours east of the meridian and near the prime vertical, observe a series of its double altitudes [§ 75] with the sextant and sidereal chronometer, and let the mean of the chronometer times be θ'. When the star reaches the same altitude west of the meridian observe its double altitudes with the vernier of the sextant set at the same readings as before, in inverted order, and let the mean of the chronometer times be θ''. The chronometer time of the star's meridian passage is therefore $\frac{1}{2}(\theta' + \theta'')$. The sidereal time of the star's meridian passage equals its right ascension a. The chronometer correction $\Delta\theta$ at this instant is given by

$$\Delta\theta = a - \tfrac{1}{2}(\theta' + \theta''). \qquad (156)$$

If a mean time chronometer is employed the sidereal time a must be converted into mean time. The required chronometer correction is then given by (156) as before.

Example. The following equal altitudes of *Arcturus* were observed with a sextant and sidereal chronometer at Ann Arbor, 1891 April 25. Required the chronometer correction.

Chronometer. East.	Sextant reading.	Chronometer. West.
$10^h\ 23^m\ 21^s$	$81°\ 30'\ 0''$	$17^h\ 21^m\ 47^s$
24 14	81 50 0	20 52
25 9	82 10 0	19 56
26 4	82 30 0	19 2
27 0	82 50 0	18 7
θ' 10 25 9.6		θ'' 17 19 56.8

Naut. Alm. p. 340, α	$14^h\ 10^m\ 42^s.8$
$\frac{1}{2}(\theta' + \theta'')$	13 52 33 .2
$\Delta\theta$	+ 18 9 .6

82. *By equal altitudes of the sun.* Observe as described above [§§ 75, 81] the two series of equal double altitudes of the sun before and after noon, and let the chronometer times of the east and west observations be T' and T'', a mean time-piece being used. The mean of the two times is not the chronometer time of the sun's meridian passage, since the sun's declination has changed during the interval, and a correction must be applied. To find its value let

t = half the interval between the observations $= \frac{1}{2}(T'' - T')$,
δ = the sun's declination at the observer's apparent noon,
$d\delta$ = the increment of the sun's declination in the interval t,
dt = the increment of the sun's hour angle due to the increment of the declination.

Differentiating (15), regarding δ and t as variables, and dividing by 15 to express dt in seconds of time, we have

$$dt = \left(\frac{\tan\phi}{\sin t} - \frac{\tan\delta}{\tan t}\right)\frac{d\delta}{15}, \tag{157}$$

by which amount the east and west observed times are greater than they would be if the declination were constant and equal to δ. The chronometer time of the sun's meridian passage is therefore $\frac{1}{2}(T' + T'') - dt$. The mean time of the sun's meridian passage is E, the equation of time at the observer's apparent noon; and therefore the chronometer correction at the mean of the two times is

$$\Delta T = E - \tfrac{1}{2}(T' + T'') + dt. \tag{158}$$

If a sidereal chronometer is employed the sidereal interval t is converted into the equivalent mean interval [§ 10], dt is computed from (157) as before and subtracted from the mean of the two times, and the result is the chronometer time of the sun's meridian passage. The sidereal time at this instant is equal to the sun's apparent right ascension α, and the chronometer correction is given by

$$\Delta\theta = \alpha - \tfrac{1}{2}(\theta' + \theta'') + dt. \tag{159}$$

Example. The equal double altitudes of the sun were observed as below at Ann Arbor, 1891 April 25, a sidereal chronometer being used. Find the chronometer correction.

Chronometer. East.	Sextant reading.	Chronometer. West.	Sun's limb.
$22^h\ 5^m\ 4^s$	$66°\ 46'\ 0''$	$5^h\ 42^m\ 21^s$	Upper
5 48	67 2 0	41 37	"
6 33	67 18 0	40 53	"
7 17	67 34 0	40 9	"
8 1	67 50 0	39 25	"
9 8.5	67 10 0	38 17	Lower
9 53	67 26 0	37 32	"
10 38	67 42 0	36 47.5	"
11 23	67 58 0	36 3	"
22 12 7.5	68 14 0	5 35 18.5	"
θ' 22 8 35.3		θ'' 5 38 50.3	

$\frac{1}{2}(\theta'' - \theta') = t$	$3^h\ 45^m\ 7^s.5$	$\tan\phi$	9.9587
Reduction	-36.9	$\sin t$	9.9192
Mean interval t	3 44 30.6	(1)	1.095
t	$3^h.742$	$\tan\delta$	9.3725
t	$56°\ 8'$	$\tan t$	0.1732
ϕ	42 17	(2)	0.158
From Naut. Alm. δ	$+13\ 16$	$\log[(1)-(2)]$	9.9717
$+48''.7 \times 3.742 = d\delta$	$+182''.2$	$\log d\delta$	2.2605
Sun's apparent α	$2^h\ 11^m\ 39^s.3$	$\log \frac{1}{15}$	8.8239
$\frac{1}{2}(\theta' + \theta'')$	1 53 42.8	$\log dt$	1.0561
dt	$+11.4$		
$\Delta\theta$	$+18\ \ 7.9$		

83. It may be convenient to observe the equal altitudes in the afternoon of one day and the forenoon of the next day. In this case the mean of the two observed times minus the proper value of dt is the chronometer time of the sun's lower culmination. If t is half the mean time interval between the observations it must be replaced by $180 + t = t'$ when substituting in

(157); and E in (158) and α in (159) must be increased by 12^h. The chronometer correction at midnight is then given by (158) and (159).

Example. Find the (sidereal) chronometer correction from the following equal altitude observations of the sun.

θ'	$5^h\ 33^m\ 56^s.3$	afternoon,	1891 April 24.
θ''	$22\ \ 8\ \ 35\ .3$	forenoon,	1891 April 25.

$\frac{1}{2}(\theta'' - \theta') = t$	$8^h\ 17^m\ 19^s.5$	$\tan\phi$	9.9587
Reduction	$-1\ \ 21\ .5$	$\sin t'$	9.9187_n
Mean interval t	$8\ \ 15\ \ 58\ .0$	(1)	-1.096
t	$8^h.266$	$\tan\delta$	9.3668
t	$123°\ 59'$	$\tan t'$	0.1713_n
t'	$303\ \ 59$	(2)	-0.157
ϕ	$42\ \ 17$	$\log[(1) - (2)]$	9.9727_n
δ	$+13\ \ 6$	$\log d\delta$	2.6075
$+49''.0 \times 8.266 = d\delta$	$+405''.0$	$\log\frac{1}{15}$	8.8239
$12^h + \alpha$	$14^h\ \ 9^m\ 46^s.4$	$\log dt$	1.4041_n
$\frac{1}{2}(\theta' + \theta'')$	$13\ \ 51\ \ 15\ .8$		
dt	$-25\ .4$		
$\Delta\theta$	$+18\ \ \ 5\ .2$		

dt is a solar interval, and should be reduced to sidereal, but the correction is small and for sextant work may be neglected.

84. The method of determining time by equal altitudes possesses the advantages that no corrections are applied for index error, eccentricity, refraction, parallax and semidiameter; any undetermined errors are eliminated from the result; and the latitude need not be accurately known. However, if the state of the atmosphere and the index correction are different at the two times of observation, the equal sextant readings do not correspond to equal true altitudes, and a correction must be applied. If the index correction is greater and the refraction less for the west observation than for the east, the true double altitude at the west observation is too great by the difference of the index corrections and twice the difference of the refractions, and the time of the observation must be increased by the interval required for the sextant reading to decrease that amount. This interval can be determined from the observations. Thus in the example of § 82, the index correction and refraction for the east observation were

$$I' = +3'\ 8'', \qquad r' = 1'\ 24'';$$

and for the west

$$I'' = +3'\ 21'', \qquad r'' = 1'\ 22''.$$

The true double altitude at the west observation was too great by

$$(I'' - I') + 2(r' - r'') = +17''.$$

From the observations it is seen that the sextant reading decreases $16' = 960''$ in about 44^s. If x is the correction to the time of the west observation, we have

$$960 : 17 = 44 : x,$$

from which $x = 0^s.8$. The correction to $\Delta\theta$ is $-\frac{1}{2}x = -0^s.4$, and therefore the true value of the chronometer correction is $\Delta\theta = +18^m\ 7^s.5$.

85. *By a single altitude of a star.* A series of double altitudes of a star having been observed in quick succession, let

R = the mean of the sextant readings,
θ' = the mean of the corresponding chronometer times;

and let

I = the index correction,
ε = the correction for eccentricity,
h' = the apparent altitude of the star,
z' = the apparent zenith distance of the star,
r = the refraction,
z = the true zenith distance of the star.

Then

$$2h' = R + I + \varepsilon = 2(90° - z'), \tag{160}$$

and

$$z = z' + r. \tag{161}$$

The latitude φ and the declination δ being known, the hour angle t is given by (38). The sidereal time at the instant of observation is given by $\theta = \alpha + t$, and thence the chronometer correction by

$$\Delta\theta = \theta - \theta'. \tag{162}$$

In case a mean time chronometer is used, the sidereal time θ must be converted into the mean time T and compared with the chronometer time T'.

Example. The observations made on *Arcturus* east of the meridian, recorded in § 81, give

$$\theta' = 10^h\ 25^m\ 9^s.6, \qquad R = 82°\ 10'\ 0''.$$

Find the chronometer correction.

R	82°	10′	0″
I		+3	12
ε		—	14
$2h'$	82	12	58
h'	41	6	29
z'	48	53	31
From (80), r		1	4
z	48	54	35
ϕ	42	16	47
Naut. Alm., δ	+19	44	52
$\phi - \delta$	22	31	55
$\phi + \delta$	62	1	39
$z + (\phi - \delta)$	71	28	30
$z - (\phi - \delta)$	26	20	40
$z + (\phi + \delta)$	110	56	14
$z - (\phi + \delta)$	−13	7	4

Barom.	29.100 inches.
Att. Therm.	55°.0 F.
Ext. Therm.	50 .0 F.
$\sin \frac{1}{2} [z + (\phi - \delta)]$	9.76646
$\sin \frac{1}{2} [z - (\phi - \phi)]$	9.35770
$\sec \frac{1}{2} [z + (\phi + \delta)]$	0.24652
$\sec \frac{1}{2} [z - (\phi + \delta)]$	0.00285
$\tan^2 \frac{1}{2} t$	9.37353
$\tan \frac{1}{2} t$	9.68676_n
$\frac{1}{2} t$	154° 4′ 26″
t	308 8 52
t	$20^h\ 32^m\ 35^s.5$
Naut. Alm., α	14 10 42 .7
θ	10 43 18 .2
θ'	10 25 9 .6
$\Delta\theta$	+18 8 .6

86. *By a single altitude of the sun.* If

p = the parallax of the sun,
S = the semidiameter of the sun,

the true zenith distance of the centre of the sun is given by

$$z = z' + r - p \pm S; \tag{163}$$

S being + or — according as the upper or lower limb of the sun was observed. The value of t is given by (38) as before. t is the true time when the observation was made. The mean time T is given by applying the equation of time E. If T' is the chronometer time of observation, the chronometer correction is

$$\Delta T = T - T'. \tag{164}$$

If a sidereal time-piece is used the mean time T must be converted into the sidereal time and the resulting value compared with the chronometer time.

Since the declination of the sun is changing, it is necessary to know the chronometer correction within 10^s; otherwise the value of δ taken from the Nautical Almanac will be slightly in error, thus giving only an approximate value of the chronometer correction. With this value of the chronometer correction a more accurate value of δ could be found, which substituted in (38) as before would give practically exact values of t and the chronometer correction.

Example. The observations made on the sun east of the meridian, recorded in § 82, give

$$\theta' = 22^h\ 8^m\ 35^s.3, \qquad R = 67°\ 30'\ 0''.$$

The chronometer correction is assumed to be $+\ 18^m\ 3^s$; required its value furnished by the observations.

R	67° 30′ 0″	Barom. 29.036 inches.		
I	+ 3 8	Att. Therm. 50°.0 F.		
ε	— 12	ExF. Therm. 47 .8 F.		
$2h'$	67 32 56			
h'	33 46 28	Naut. Alm., p. 278, π		8″.8
z'	56 13 32	log π		0.944
r	1 24	sin z'		9.920
p	7	From (57), p		7″
z	56 14 49			
ϕ	42 16 47	θ		$22^h\ 8^m\ 35^s$
δ	+ 13 12 53	Approx. $\Delta\theta$		+ 18 3
$\phi - \delta$	29 3 54	Sid. time		22 26 38
$\phi + \delta$	55 29 40	Mean time	April 24^d	20 13 29
$z + (\phi - \delta)$	85 18 43	Longitude		5 34 55
$z - (\phi - \delta)$	27 10 55	Gr. mean time	April 25	1 48 24
$z + (\phi + \delta)$	111 44 29	Naut. Alm., δ		+ 13° 12′ 53″
$z - (\phi + \delta)$	0 45 9			
$\sin \frac{1}{2}[z + (\phi - \delta)]$	9.83097	True time	April 24^d	$20^h\ 15^m\ 38^s.3$
$\sin \frac{1}{2}[z - (\phi - \delta)]$	9.37104	Longitude		5 34 55 .1
$\sec \frac{1}{2}[z + (\phi + \delta)]$	0.25099	Gr. true time	April 25	1 50 33 .4
$\sec \frac{1}{2}[z - (\phi + \delta)]$	0.00001	Eq. of time, E		— 2 4 .9
$\tan^2 \frac{1}{2} t$	9.45301	Mean time	April 24	20 13 33 .4
$\tan \frac{1}{2} t$	9.72650_n	Sid. time, θ		22 26 42 .4
$\frac{1}{2} t$	151° 57′ 17″	Chron. time, θ'		22 8 35 .3
t	303 54 34	$\Delta\theta$		+ 18 7 .1
True time	$20^h\ 15^m\ 38^s.3$			

This value of $\Delta\theta$ differs so little from the assumed value that another approximation to the value of δ is unnecessary.

The value of $\Delta\theta$ given by the afternoon solar observations, § 82, is $+\ 18^m\ 8^s.1$, which agrees well with the above, assuming the chronometer's daily rate to be $+\ 3^s.6$ [§ 66].

87. The error in the hour angle—and therefore in the time—produced by a small error in the measured altitude or in the assumed latitude is readily found. Differentiating (15), regarding z and t as variables, and reducing by (17), we obtain

$$dt = \frac{dz}{\sin A \cos\phi}; \tag{165}$$

that is, an error dz in the measured zenith distance produces an error dt in the time, which is least when sin A is a maximum. Likewise, differentiating (15) with respect to φ and t and reducing by (16) and (17), we have

$$dt = -\frac{d\phi}{\tan A \cos\phi}; \tag{166}$$

that is, an error $d\varphi$ in the latitude gives rise to an error dt in the time, which is small when tan A is large.

For these reasons it appears that to obtain the best determination of time from observed altitudes, those stars should be selected which are as nearly as possible in the prime vertical.

GEOGRAPHICAL LATITUDE.

88. *By a meridian altitude of a star or the sun.* Observe the double altitude of the star or sun at the instant when it is on the meridian, and obtain the true zenith distance z as in §§ 85 and 86. The latitude is then found from

$$\phi = \delta \pm z, \tag{167}$$

the upper sign being used for a star south of the zenith, the lower sign for a star between the zenith and the pole. For a star below the pole we have

$$\varphi = 180^\circ - \delta - z. \tag{168}$$

Example. The double altitude of the sun's lower limb was observed at Ann Arbor at true noon 1891 Feb. 6, as follows:

Sextant 63° 49′ 15″, Barom. 28.98 inches, Ext. Therm. 38° F.

Find the latitude.

R	63°	49′	15″		z'	58°	3′	56″
I	+	3	5		r		1	28
ε	—		12		p			8
$2h'$	63	52	8		S		— 16	15
h'	31	56	4		z	57	49	1
z'	58	3	56		δ	— 15	32	11
					ϕ	42	16	50

89. *By an altitude of a star, the time being known.* Having determined the star's hour angle by (40), the latitude is given

by (15), in which φ is the only unknown quantity. To determine it, assume

$$f \sin F = \cos \delta \cos t, \tag{169}$$

$$f \cos F = \sin \delta, \tag{170}$$

and (15) becomes

$$\cos z = f \sin (\phi + F) = \sin \delta \sec F \sin (\phi + F).$$

From these we obtain

$$\tan F = \cot \delta \cos t, \tag{171}$$

$$\sin (\phi + F) = \cos F \cos z \operatorname{cosec} \delta, \tag{172}$$

which effect the solution.

The quadrant of F is determined by (169) and (170). $(\varphi + F)$, being determined from its sine, may terminate in either of two quadrants, thus giving rise to two values of the latitude. That one is selected which agrees best with the known approximate value of the latitude.

In case the sun is observed, t is the true solar time.

Example. Find the latitude from the following double altitudes of *Polaris* observed 1891 April 25:

	Chronometer.		Sextant.		
	$14^h\ 55^m\ 5^s$		82° 18′ 40″	Barom.	29.17 inches.
	56 35		19 20	Ext. Therm.	39°.3 F.
	58 20		19 35		Naut. Alm. p. 305,
	15 1 0		20 40	α	$1^h\ 17^m\ 48^s$
Means	14 57 45		82 19 34	δ	88° 43′ 32″
θ'	$14^h\ 57^m\ 45^s$	R	82° 19′ 34″	$\cot \delta$	8.347270
$\Delta\theta$	+ 18 10	I	+ 3 12	$\cos t$	9.939572_n
θ	15 15 55	ε	— 15	F	358° 53′ 28″
α	1 17 48	$2h'$	82 22 31	$\cos F$	9.999919
t	13 58 7	h'	41 11 15	$\cos z$	9.818414
t	209° 31′ 45″	z'	48 48 45	$\operatorname{cosec} \delta$	0.000108
		r	1 6	$\sin (\phi + F)$	9.818441
		z	48 49 51	$\phi + F$	41° 10′ 20″
				ϕ	42 16 52

90. Differentiating (15) with regard to z and φ and reducing by (16) we obtain

$$d\phi = \frac{dz}{\cos A}; \tag{173}$$

that is, an error dz in the measured zenith distance produces the minimum error $d\varphi$ in the latitude when the star is on the meridian.

Differentiating (15) with regard to φ and t and reducing by (16) and (17) we obtain

$$d\phi = -\tan A \cos\phi \, dt; \tag{174}$$

that is, an error dt in the estimated time of making the observation gives rise to an error $d\varphi$ in the latitude, which will be small when the star is nearly on the meridian, and equal to zero when A is $0°$ or $180°$.

For these reasons it appears that to obtain the best determination of the latitude from observed altitudes, those stars should be selected which are as nearly as possible on the meridian.

91. *By circummeridian altitudes.* The method of § 88 is applicable to only one altitude observed when the star is on the meridian. If a series of altitudes be observed just before and after meridian passage,—called *circummeridian altitudes*,—they can be reduced to the equivalent meridian altitudes and a very approximate value of the latitude obtained by combining the results. Equation (15) may be written

$$\cos z = \cos(\phi - \delta) - \cos\delta \cos\phi \, 2\sin^2 \tfrac{1}{2} t. \tag{175}$$

If we let z_0 be the zenith distance of the star when it is on the meridian and put $y = \cos\delta \cos\varphi \, 2\sin^2 \frac{1}{2} t$, (175) becomes

$$\cos z = \cos z_0 - y. \tag{176}$$

Here z is a function of y, and we may write

$$z = f(y).$$

Developing this in series by Maclaurin's formula, restoring the value of y, and dividing the abstract terms by $\sin 1''$ to express them in seconds of arc, we have

$$z = z_0 + \frac{\cos\varphi \cos\delta}{\sin z_0} \cdot \frac{2\sin^2 \frac{1}{2} t}{\sin 1''} - \left(\frac{\cos\varphi \cos\delta}{\sin z_0}\right)^2 \cdot \frac{\cot z_0 \, 2\sin^4 \frac{1}{2} t}{\sin 1''} + \cdots, \tag{177}$$

which converges rapidly when t does not exceed 30^m, and the star is more than $20°$ from the zenith, as it will be in sextant double altitudes. If we let

$$\frac{\cos\phi \cos\delta}{\sin z_0} = A, \qquad A^2 \cot z_0 = B, \tag{178}$$

$$\frac{2\sin^2 \frac{1}{2} t}{\sin 1''} = m, \qquad \frac{2\sin^4 \frac{1}{2} t}{\sin 1''} = n, \tag{179}$$

and substitute the resulting value of z_0 for z in (167), we have

$$\phi = \delta \pm z \mp Am \pm Bn, \tag{180}$$

the lower sign being employed for a star culminating between the zenith and the pole.

When a star is observed near the meridian at lower culmination, it is convenient to reckon the hour angle from the lower transit. t in (15) must be replaced by $180° + t$, and we obtain

$$\cos z = \sin\delta \sin\phi - \cos\delta \cos\phi \cos t = -\cos(\varphi + \delta) + \cos\delta \cos\phi\, 2\sin^2 \tfrac{1}{2} t.$$

Developing this in series as before and substituting the resulting value of z_0 for z in (168), we have

$$\phi = 180° - \delta - z - Am - Bn. \tag{181}$$

The entire series of observations is conveniently reduced as a single observation by letting z, m and n in (180) and (181) represent the arithmetical means of the values of these quantities for the individual observations.

The values of m and n are tabulated in the Appendix, TABLE III, with the argument t.

An approximate value of φ is required in computing A. This may be obtained by the method of § 88, from the observation made nearest the meridian.

If the sun is observed the declination is taken from the Nautical Almanac for the instant of each observation in case the observations are reduced separately, and for the mean of the times in case they are reduced collectively.

If a star is observed with a sidereal chronometer the hour angles t are the intervals between each observed time and the chronometer time of the star's transit.

If a star is observed with a mean time chronometer, the intervals must be reduced from mean to sidereal intervals before entering TABLE III for m and n.

If the sun is observed with a mean time chronometer the intervals should be reduced to apparent solar intervals by correcting for the change in the equation of time during the intervals. This, however, will never exceed $0^s.5$, and may be neglected in sextant observations.

If the sun is observed with a sidereal chronometer, the intervals must be reduced to mean solar intervals and thence to apparent solar.

If the rate of the chronometer is large it must be allowed for.

Example. Wednesday, 1891 April 8, at a place in latitude about 42° 17′ and longitude $5^h\ 34^m\ 55^s$ the following double altitudes of the sun were observed with a sextant and sidereal chronometer. Barom. 29.373 inches, Att. Therm. 66° F., Ext. Therm. 42°.5 F. Required the latitude. [Each recorded observation is the mean of three consecutive original observations.]

Limb.	Sextant.	Chronometer.	Sid. t	Solar t	m	n
Upper	109° 58′ 33″.7	$0^h\ 31^m\ 28^s.0$	$-19^m\ 59^s.2$	$-19^m\ 55^s.9$	779″.6	1″.47
Lower	109 4 40 .0	34 51 .0	−16 36 .2	−16 33 .5	538 .1	0 .70
"	109 14 6 .2	38 43 .3	−12 43 .9	−12 41 .8	316 .4	0 .24
Upper	110 25 16 .2	42 42 .3	− 8 44 .9	− 8 43 .5	149 .5	0 .05
"	110 29 39 .0	45 44 .7	− 5 42 .5	− 5 41 .6	63 .6	0 .01
Lower	109 27 27 .5	49 30 .7	− 1 56 .5	− 1 56 .2	7 .4	0 .00
"	109 27 43 .7	53 9 .7	+ 1 42 .5	+ 1 42 .2	5 .7	0 .00
Upper	110 29 0 .0	0 57 36 .0	+ 6 8 .8	+ 6 7 .8	73 .8	0 .01
"	110 21 51 .2	1 2 56 .0	+11 28 .8	+11 26 .9	257 .3	0 .16
Lower	109 11 33 .7	5 46 .7	+14 19 .5	+14 17 .2	400 .6	0 .39
"	109 2 27 .7	9 12 .3	+17 45 .1	+17 42 .2	615 .0	0 .92
Upper	109 56 20 .0	1 12 33 .7	+21 6 .5	+21 3 .0	869 .4	1 .83
	109 45 43 .2			+ 0 33 .9	339 .7	0 .48

Apparent time of apparent noon	$0^h\ 0^m\ 0^s.0$
Equation of time	+ 1 52 .1
Mean time of apparent noon	0 1 52 .1
Sidereal time of apparent noon	1 8 37 .2
Chronometer correction	+17 10 .0
Chronometer time of apparent noon	0 51 27 .2

The difference between this and the observed times gives the sidereal intervals t as above.

The mean of the hour angles is $+0^m\ 33^s.9$, and therefore the sun's declination is taken for the local mean time $0^h\ 1^m\ 52^s.1 + 0^m\ 33^s.9 = 0^h\ 2^m\ 26^s.0$, or Greenwich mean time $5^h\ 37^m\ 21^s.0$.

An equal number of observations on the upper and lower limbs were made, hence there is no correction for semidiameter.

The solution of (180) is made as follows:

6

δ	$+7°\ 17'\ 33''.4$	Sextant	$109°\ 45'\ 43''.2$
ϕ	42 17	I	$+2\ 51\ .7$
z_0	34 59 26 .6	ε	$-\ 18\ .0$
$\cos\phi$	9.86913	$2h'$	109 48 16 .9
$\cos\delta$	9.99647	h'	54 54 8 .4
$\operatorname{cosec} z_0$	0.24151	z'	35 5 51 .6
$\log A$	0.10711	r'	40 .5
$\log m$	2.53110	p	5 .1
Am	$434''.7$	z	35 6 27 .0
$\log A^2$	0.2142	δ	$+7$ 17 33 .4
$\cot z_0$	0.1549	Am	7 14 .7
$\log n$	9.6812	Bn	1 .1
$\log Bn$	0 0503	ϕ	42 16 46 .8
Bn	$1''.1$		

A repetition of the computation with this value of φ does not change the result.

[The latitude of the place is known to be about $42°\ 16'\ 47''.1$].

GEOGRAPHICAL LONGITUDE.

92. *By lunar distances.* The moon's distance from a star nearly in the ecliptic is rapidly changing. Its geocentric distances from the sun, Venus, Mars, Jupiter, Saturn and nine bright stars near its path are given in the Nautical Almanac [pp. XIII -XVIII of each month] at three-hour intervals of Greenwich mean time, from which the distances at any other instants may be found by interpolation. Conversely, if its distance from any of these objects is measured with the sextant and the apparent distance reduced to the corresponding geocentric distance, the Greenwich mean time at the instant of observation can be found. This minus the observer's mean time is the observer's longitude. The method is of great importance to navigators and explorers.

93. We shall suppose that the moon's distance *from the sun* has been observed. The formulæ for a planet will be the same, save that the semidiameter of the planet may usually* be neglected. For a star the parallax and semidiameter are zero.

The sextant reading having been corrected for the index error and eccentricity, the result is the apparent distance between the nearest limbs of the sun and moon. It must be corrected for their semidiameters, refractions and parallaxes.

To compute these corrections, the zenith distances of the two bodies must be known. When there are three observers, as

* In case the telescope is powerful enough to define the planet's disk, the moon's limb may be made to pass through the center of the disk.

frequently happens at sea, the altitudes of the sun and moon, and the distance between them, should be measured simultaneously. The observer's mean time can be obtained also from these observed altitudes of the sun [§ 86]. When it is not practicable to make these observations at the same time, the observer may measure the altitudes immediately before and after measuring the lunar distance, and obtain the required altitudes at the instant of observation by interpolation. Again, the observer may *assume an approximate value of the longitude* (which he can usually do sufficiently accurately), and take from the Nautical Almanac the right ascensions and declinations of the sun and moon corresponding to the Greenwich time thus obtained. The hour angles, azimuths and zenith distances are then given by §§ 18 and 14.

The parallax of the sun in azimuth is negligible; its parallax in zenith distance is given by (57). The parallax of the moon in azimuth is given by (64) and (65); and in zenith distance, by (73), (71) and (72). (80) gives the refractions, care being taken to use the *apparent* zenith distance. The semidiameters of the sun and moon are obtained by the methods of §§ 33–35. The solution of (91) requires the values of q. In Fig. 13 let M be the moon's centre, S the sun's centre, and Z the zenith. For the sun, $q = ZSM$, and for the moon, $q = ZMS$. If we let

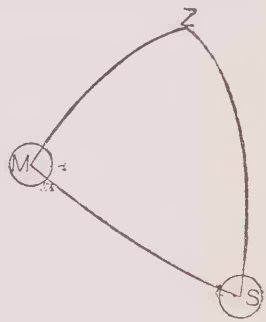

Fig. 13.

Z' = the apparent zenith distance of the sun = ZS,
z' = the apparent zenith distance of the moon = ZM,
d' = the apparent distance between the centres = SM,

we can write, for the sun,

$$\tan \tfrac{1}{2} q = \sqrt{\frac{\sin \frac{1}{2}(z' - Z' + d') \sin \frac{1}{2}(z' + Z' - d')}{\sin \frac{1}{2}(Z' + z' + d') \sin \frac{1}{2}(Z' - z' + d')}}; \qquad (182)$$

and for the moon,

$$\tan \tfrac{1}{2} q = \sqrt{\frac{\sin \frac{1}{2}(Z' - z' + d') \sin \frac{1}{2}(Z' + z' - d')}{\sin \frac{1}{2}(z' + Z' + d') \sin \frac{1}{2}(z' - Z' + d')}}. \qquad (183)$$

Adding the inclined semidiameters given by (91) to the corrected sextant reading, the sum is the distance d' between the centres as seen from the observer.

The combined effect of the refraction and the parallax in zenith distance is to shift the bodies in their vertical circles without changing the angle SZM at the zenith, which we shall represent by V. If we let

$Z =$ the geocentric zenith distance of the sun,
$z =$ the geocentric zenith distance of the moon,
$d'' =$ the corresponding distance between the centres,

we can write

$$\cos d'' = \cos z \cos Z + \sin z \sin Z \cos V, \tag{184}$$
$$\cos d' = \cos z' \cos Z' + \sin z' \sin Z' \cos V. \tag{185}$$

Therefore

$$\frac{\cos d'' - \cos z \cos Z}{\sin z \sin Z} = \frac{\cos d' - \cos z' \cos Z'}{\sin z' \sin Z'};$$

or

$$\frac{\cos d'' - \cos (z + Z)}{\sin z \sin Z} = \frac{\cos d' - \cos (z' + Z')}{\sin z' \sin Z'}. \tag{186}$$

If we put $z' + Z' + d' = 2x$, and substitute

$$\cos d' - \cos (z' + Z') = 2 \sin x \sin (x - d'),$$
$$\cos d'' = 1 - 2 \sin^2 \tfrac{1}{2} d'',$$
$$\cos (z + Z) = 2 \cos^2 \tfrac{1}{2} (z + Z) - 1,$$

(186) reduces to

$$\sin^2 \tfrac{1}{2} d'' = \sin^2 \tfrac{1}{2} (z + Z) - \frac{\sin z \sin Z}{\sin z' \sin Z'} \sin x \sin (x - d'). \tag{187}$$

Let an auxiliary angle M be defined by

$$\sin^2 M = \frac{\sin z \sin Z}{\sin z' \sin Z'} \cdot \frac{\sin x \sin (x - d')}{\sin^2 \frac{1}{2} (z + Z)}. \tag{188}$$

(187) takes the form

$$\sin \tfrac{1}{2} d'' = \sin \tfrac{1}{2} (z + Z) \cos M. \tag{189}$$

The parallax of the moon in azimuth produces a small change in V and therefore in d''. From (184), by differentiation,

$$\Delta d'' = \sin z \sin Z \sin V \operatorname{cosec} d'' \, \Delta V, \tag{190}$$

in which ΔV is the parallax in azimuth.

The geocentric distance d between the centres is now given by

$$d = d'' + \Delta d'' \tag{191}$$

In connection with the lunar distances, the Nautical Almanac gives a column "P. L. of Diff." (Proportional Logarithm of the

Difference), which is the logarithm of 10800, the number of seconds in 3^h, minus the logarithm of the change in the lunar distance, expressed in seconds of arc, in the next following three hours. That is, it is the logarithm of the reciprocal of the moon's *average rate* for the three hours, or the rate at the middle period of the three hours [see remarks on interpolation, § 9]. In order to interpolate for the Greenwich mean time corresponding to the given value of d we have only to add the P. L. of Diff. for the middle period of the approximate interval to the logarithm of the number of seconds of arc by which d exceeds the next smaller Nautical Almanac lunar distance. The sum is the logarithm of the number of seconds of time by which the Nautical Almanac time is to be increased.

If the P. L. of Diff. given in the Nautical Almanac is used without change, a slight correction for neglected second difference of the moon's rate can be taken from TABLE I, Appendix, Nautical Almanac, and applied as there directed.

If the resulting longitude differs considerably from the assumed longitude, a second approximation should be made by starting with the value of the longitude just obtained. A third approximation will not be necessary.

Example. 1891 May 12, the distance between the bright limbs of the sun and moon was observed with a sextant and sidereal chronometer. The mean of ten observations gave

$$\theta' = 8^h\ 36^m\ 10^s, \qquad R = 57^\circ\ 28'\ 32''.9.$$

Chronometer correction, $+\ 19^m\ 15^s$; index correction, $+\ 2'\ 56''.4$; Barom. 29.25 inches, Att. Therm. 62° F., Ext. Therm. 57° F.; latitude, $+\ 42^\circ\ 16'\ 47''$; longitude *assumed*, $+\ 5^h\ 34^m$. Required a more exact value of the longitude.

θ'	$8^h\ 36^m\ 10^s$	R	$57^\circ\ 28'\ 32''.9$
$\Delta\theta$	$+\ 19\ \ 15$	I	$+\ \ 2\ \ 56\ .4$
θ	$8\ \ 55\ \ 25$	ε	$-\ \ \ \ 11\ .3$
Mean time	5 33 42	Distance	57 31 18 .0
Longitude	5 34		
Gr. mean time	11 7 42		

Corresponding to this Greenwich mean time, we take from the Nautical Almanac, pp. 74, 75, 77, 80 and 278,

		Sun.	Moon.
Right ascension,	α	$3^h\ 17^m\ 54^s$	$7^h\ 29^m\ 31^s$
Declination,	δ	$+18^\circ\ 15'\ 10''$	$+25^\circ\ 36'\ 55''$
Semidiameter,	S	15 51.7	15 12.8
Horizontal parallax,	π	8.8	55 43.3

By §§ 18 and 14 we find for the geocentric co-ordinates of the sun,

$$t = 84^\circ\ 22'\ 45'', \qquad A = 100^\circ\ 8'\ 50'', \qquad Z = 73^\circ\ 46'\ 5'';$$

and for the moon,

$$t = 21^\circ\ 28'\ 30'', \qquad A = 53^\circ\ 27'\ 21'', \qquad z = 24^\circ\ 15'\ 40''.$$

Computing the parallaxes we obtain, for the sun,

$$A' - A = 0, \qquad p = Z' - Z = 8''.4.$$

From (53) we find $\varphi - \varphi' = 687''.3 = 11'\ 27''.3$; and from (52) $\log \rho = 9.99935$; therefore, for the moon,

$$\log m = 6.11807, \qquad A' - A = +21''.8,$$
$$\gamma = +6'\ 49''.2, \qquad \log n = 8.20908, \qquad z' - z = 23'\ 6''.2.$$

The mean refraction of the sun, TABLE II, is about 3′ 13″, and therefore its apparent zenith distance is very nearly 73° 43′ 0″. The exact value of the refraction is now found from (80) to be 3′ 8″. 6. Similarly, the refraction for the moon is 25″.6. The apparent zenith distances of the sun and moon are therefore

$$Z' = 73^\circ\ 43'\ 4''.8, \qquad z' = 24^\circ\ 38'\ 20''.6.$$

The apparent zenith distance of the upper limb of the sun is 73° 43′ 4″.8 — 15′ 51″.7 = 73° 27′ 13″.1. The corresponding refraction is 3′ 5″.5. The apparent vertical semidiameter is therefore contracted 3″.1 [§ 35], and its value is 15′ 48″.6.

The moon's apparent diameter is found from (90) to be 15′ 26″.3; and by refraction its apparent vertical semidiameter is reduced to 15′ 26″.0.

The approximate distance between the centres of the sun and moon is

$$d' = 57^\circ\ 31'\ 18'' + 15'\ 49'' + 15'\ 26'' = 58^\circ\ 2'\ 33''.$$

Substituting these values of d', Z' and z' in (182) we obtain for the sun, $q = 20^\circ\ 58'$; and in (183) for the moon, $q = 124^\circ\ 34'$. For the sun, $a = 15'\ 51''.7 = 951''.7$, $b = 15'\ 48''.6 = 948''.6$; and by (91) the inclined semidiameter is $S'' = 15'\ 49''.1$. Similarly, for the moon, $S'' = 15'\ 26''.2$. The apparent distance between the centres of the sun and moon is therefore

$$d' = 57^\circ\ 31'\ 18''.0 + 15'\ 49''.1 + 15'\ 26''.2 = 58^\circ\ 2'\ 33''.3.$$

The solution of (188) gives $M = 49^\circ\ 48'\ 39''.0$; and thence, from (189), $d'' = 58^\circ\ 18'\ 16''.0$. Substituting $A' - A = \Delta V = +21''.8$ in (190), we obtain $\Delta d'' = +7''.4$. The geocentric distance between the sun and moon is therefore, by (191),

$$d = 58°\ 18'\ 16''.0 + 7''.4 = 58°\ 18'\ 23''.4.$$

From the Nautical Almanac, pp. 86 and 87, at

Greenwich mean time 9^h,	$d = 57°\ 16'\ 23''$,	P. L. of Diff. = 0.3169,
" " " 12,	$d = 58\ \ 43\ \ \ 9$,	P. L. of Diff. = 0.3184.

We have to interpolate for the interval of time T after 9^h, corresponding to a change in d of $58°\ 18'\ 23'' - 57°\ 16'\ 23'' = 3720''$. The value of T is approximately 2^h. The value of P. L. of Diff. at the middle of the 2^h is 0.3167.

P. L. of Diff.	0.3167	Gr. mean time	11^h	8^m	32^s
log 3720	3.5705	Observer's mean time	5	33	42
log T	3.8872	Observer's longitude	5	34	50
T	7712^s				
T	$2^h\ 8^m\ 32^s$				

The true value of the longitude is known to be $5^h\ 34^m\ 55^s$. The error of 5^s corresponds to an error of $3''$ in the measured distance [or in the lunar tables], and is unusually small. The observations are difficult to make, and the measures of the best observers are easily liable to an error of $10''$. It is well, however, to carry the numerous corrections to tenths of a second to prevent the accumulated effect of neglected fractions.

CHAPTER VIII.

THE TRANSIT INSTRUMENT.

94. The transit instrument consists essentially of a telescope attached perpendicularly to a horizontal axis. The cylindrical extremities of this axis are the *pivots*. The straight line passing through their centres is the *rotation axis*. The straight line passing through the optical centre of the object glass and the rotation axis and perpendicular to the latter is the *collimation axis*. By revolving the instrument about the rotation axis the collimation axis describes a plane called the *collimation plane*. In the common focus of the object glass and eye piece is a system of wires called the *reticule*. It consists either of spider threads attached to a frame, or of fine lines ruled on thin glass. An odd number of wires—usually five or seven—is placed parallel to the collimation plane and perpendicular to the collimation axis, over which the times of transit of a star's image are observed. The *middle wire* of the set is fixed as nearly as possible in the collimation plane. One or two wires are placed perpendicular to these to mark the centre of the field of view. A micrometer

wire parallel to the first set is usually arranged to move as nearly as possible in their plane. The axis of the instrument is hollow. A light is placed so that the rays from it enter the axis and fall on a small mirror in the centre of the telescope which reflects them to the eye piece in such a way that the wires are seen as dark lines in a bright field.

The instrument is so arranged that it can be rotated 180°, *i. e.*, *reversed*, about a vertical line. The two positions are defined conveniently by stating the position of the clamp on the axis. Thus. *clamp W* or *clamp E* denotes that position of the instrument in which the clamp is west or east of the collimation plane.

Another common form of this instrument is that in which one end of the axis is made to take the place of the lower half of the telescope. A prism is placed at the intersection of the telescope and axis which turns the rays of light through 90° to the eye piece, which is in one end of the axis. This form is sometimes called the *broken* or *prismatic transit*. Other forms are used, but they require no special description or theory.

95. The transit instrument is mounted so that its collimation plane is either very nearly in the prime vertical, or very nearly in the meridian. In the first case it can be used to determine the latitude; but this method is practically superseded by that of the zenith telescope, to be described later. Mounted in the meridian it is employed in connection with a sidereal clock or chronometer to determine the time and longitude when great accuracy is required, and we shall treat only this case.

Let us suppose that the axis is mounted due east and west and that the middle wire is exactly in the collimation plane. If the image of a star whose apparent right ascension is a is observed on the wire at the chronometer time θ', the chronometer correction $\Delta\theta$ is given by (neglecting diurnal aberration)

$$\Delta\theta = a - \theta'. \qquad (192)$$

The observer may adjust his instrument as accurately as he pleases, but the adjustments will not remain, owing to changes of temperature, strains, etc. It is customary to put the instrument very nearly in the meridian when it is first set up, and thereafter to vary its parts only at long intervals of time. In general, therefore, the star will be observed when it is slightly

to one side of the meridian. A determination of the errors of adjustment of his instrument enables the observer to reduce the chronometer time of observation to the chronometer time of meridian passage; whence the chronometer correction is given by (192) as before.

96. The rotation axis should be in the prime vertical and in the horizon, and the middle wire should be in the collimation plane.

The *azimuth constant*, a, is the angle which the rotation axis makes with the prime vertical. It is + when the west end of the axis is too far south.

The *level constant*, b, is the angle which the rotation axis makes with the horizon. It is + when the west end of the axis is too high.

The *collimation constant*, c, is the angle which a line through the middle wire and the optical centre of the object glass—called the line of sight—makes with the collimation plane. It is + when the middle wire is west of the collimation plane.

It is required to correct the time of observation of a star for the small deviations a, b and c.

Let $SWNE$ in Fig. 14 represent the celestial sphere projected on the horizon, Z the observer's zenith, NS the meridian, WE the prime vertical, WQE the equator, and P the pole. Suppose the rotation axis of the instrument lies in the vertical circle AZB, and that the axis produced cuts the sphere in A and B; that the great circle $N'Z'S'$ lies in the collimation plane; and that $N''Z''S''$, parallel to $N'Z'S'$, is described by the line through the middle wire and the centre of the object glass. When the stars are observed on the middle wire they are on the circle $N''Z''S''$, whereas we desire to know the chronometer time when they are on the meridian. Let O be such a star. The time required for the star to pass from O to the meridian is

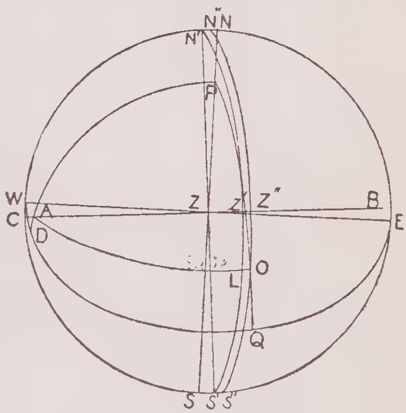

Fig. 14.

equal to the hour angle of O measured from the meridian *toward the east.* Let τ represent it.

If we let $90 - m$ denote the hour angle and n the declination of A, we have by definition,

$$\begin{aligned} ZPA &= 90^\circ - m, & ZA &= 90^\circ - b, \\ PZA &= 90^\circ + a, & PO &= 90^\circ - \delta, \\ PA &= 90^\circ - n, & AO &= 90^\circ + c, \\ PZ &= 90^\circ - \phi, & OPA &= 90^\circ - m + \tau. \end{aligned}$$

From the triangle ZPA we have

$$\sin n = \sin b \sin \phi - \cos b \cos \phi \sin a, \tag{193}$$

$$\cos n \sin m = \sin b \cos \phi + \cos b \sin \phi \sin a; \tag{194}$$

and from OPA

$$\sin c = - \sin n \sin \delta + \cos n \cos \delta \sin (\tau - m),$$

or

$$\sin (\tau - m) = \tan n \tan \delta + \sin c \sec n \sec \delta. \tag{195}$$

These equations are true for any position of the instrument, and determine τ when a, b and c are known. But for the instrument nearly in the meridian a, b, c, m and n are small, and the above equations become

$$n = b \sin \phi - a \cos \phi, \tag{196}$$

$$m = b \cos \phi + a \sin \phi, \tag{197}$$

$$\tau = m + n \tan \delta + c \sec \delta. \tag{198}$$

(198) is Bessel's formula for computing the value of τ. Eliminating m and n from the three equations we obtain Mayer's formula

$$\tau = a \,.\, \frac{\sin (\phi - \delta)}{\cos \delta} + b \,.\, \frac{\cos (\phi - \delta)}{\cos \delta} + c \,.\, \frac{1}{\cos \delta}, \tag{199}$$

in which the terms of the second member are the corrections, respectively, for errors of adjustment in azimuth, level and collimation.

For convenience, let us put

$$A = \frac{\sin (\phi - \delta)}{\cos \delta}, \qquad B = \frac{\cos (\phi - \delta)}{\cos \delta}, \qquad C = \frac{1}{\cos \delta}, \tag{200}$$

and (199) becomes

$$\tau = aA + bB + cC. \tag{201}$$

The effect of the diurnal aberration is to throw the star east of its true position. It is therefore observed too late, and the time of observation must be diminished by the quantity, (100),

$$0''.31 \cos\phi \sec\delta = 0^s.021 \cos\phi\ C. \tag{202}$$

For greater accuracy the star is observed over several wires. An odd number of wires is always used. They are generally placed very nearly equidistant, or very nearly symmetrical with respect to the middle wire. Were either of these arrangements exactly realized the mean of all the times of transit would be the most probable time of transit over the middle wire. This never happens, however, and it is necessary to determine the intervals between the wires.

Let i denote the angular distance between a side wire and the middle wire;* I the interval of time required by a star whose declination is δ to pass through this distance. From Fig. 11, letting the two positions of the micrometer wire represent the side wire and the middle wire, we have in the triangle $CS'P$,

$$CS' = i, \qquad S'P = 90° - \delta, \qquad CPS' = I;$$

and we can write

$$\sin I = \sin i \sec\delta = \sin i\ C. \tag{203}$$

If the star is not within 10° of the pole, it is sufficiently accurate to use

$$I = i \sec\delta = i\ C. \tag{204}$$

Suppose there are five threads in the reticule, numbered I, II, III, IV, V, beginning on the side next to the clamp, and that the clamp is west. Let t_1, t_2, t_3, t_4, t_5, be the observed times of transit of the star over the wires, and i_1, i_2, i_4, i_5, the distances of the four side wires from the middle wire.† The five observed transits give for the time of crossing the middle wire either $t_1 + i_1 C$, $t_2 + i_2 C$, t_3, $t_4 + i_4 C$, or $t_5 + i_5 C$, which would all be equal if the observations were perfect. Taking their mean, the most probable time of crossing the middle wire is

$$\frac{t_1 + t_2 + t_3 + t_4 + t_5}{5} + \frac{i_1 + i_2 + i_4 + i_5}{5} C.$$

If we let

$$t_m = \frac{t_1 + t_2 + t_3 + t_4 + t_5}{5}, \tag{205}$$

$$i_m = \frac{i_1 + i_2 + i_4 + i_5}{5}, \tag{206}$$

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* That is, the angle subtended at the optical centre of the object glass by lines drawn to the side wire and to the middle wire. It s also the interval of time required for a star in the equator to pass from the side wire to the middle wire.

† For clamp west i_4 and i_5 are negative; for clamp east i_1 and i_2 are negative.

the most probable time of crossing the middle wire is

$$\theta_m + i_m C. \tag{207}$$

θ_m is the time of crossing a fictitious wire called the *mean wire*, and $i_m C$ is the reduction from the mean wire to the middle wire.

The above method holds good also in the case of an *incomplete* transit; that is, one in which the transits over some of the wires have been missed. Thus, suppose the wires I and IV have been missed. The three remaining transits give for the times of crossing the middle wire $t_2 + i_2 C$, t_3, $t_5 + i_5 C$; and their mean is

$$\frac{t_2 + t_3 + t_5}{3} + \frac{i_2 + i_5}{3} C, \tag{208}$$

and similarly in other cases.

In accurate determinations of the time several stars will be observed, and if the chronometer has a sensible rate, the chronometer corrections at the several times of observation will be different. To equalize them, let θ_0 be some chronometer time near the middle of the series of observations, let a star be observed at the time θ_m, and let the rate of the chronometer be $\delta\theta$. During the interval $\theta_m - \theta_0$ the chronometer loses

$$(\theta_m - \theta_0)\,\delta\theta. \tag{209}$$

If this quantity be computed for all the stars observed and applied to the observed times, the resulting chronometer corrections furnished by the several stars will be the corrections at the instant θ_0, and with perfect observations would all be equal.

Collecting the expressions (201), (202), (207) and (209), we have for the observed time of crossing the meridian when the clamp is west,

$$\theta' = \theta_m + aA + bB + cC - 0^s.021 \cos\phi\, C + i_m C + (\theta_m - \theta_0)\,\delta\theta; \tag{210}$$

and therefore, by (192),

$$\Delta\theta = a - [\theta_m + aA + bB + (c - 0^s.021 \cos\phi + i_m)\, C + (\theta_m - \theta_0)\,\delta\theta]. \tag{211}$$

For clamp east it is easily seen that c and i_m change sign; otherwise the formula remains the same.

The formula has been deduced for a star observed at upper culmination. For a star observed at lower culmination we have only to replace δ by $180° - \delta$ in the factors A, B and C, and they become

$$A = \frac{\sin(\phi + \delta)}{\cos \delta}, \quad B = \frac{\cos(\phi + \delta)}{\cos \delta}, \quad C = -\frac{1}{\cos \delta}. \qquad (212)$$

The factors A, B and C, are readily computed with four-place tables. But when an instrument is set up permanently, as in an observatory, their values should be computed for every degree of declination and tabulated. For polar distances less than 15° it is convenient to have them tabulated for every ten minutes of declination.

DETERMINATION OF THE WIRE INTERVALS.

97. (a). If the instrument is provided with a micrometer in right ascension, set the micrometer wire in succession on each of the fixed wires.* The differences of the micrometer readings on the side wires and the middle wire give the intervals in terms of one revolution of the screw, which has been obtained by § 60.

Example. 1891 Feb. 20. The transit instrument of the Detroit Observatory. Four sets of micrometer readings are given when the micrometer wire was in contact with each side of the fixed wires, to find the wire intervals and i_m. The numbers in the last line are the means of all the readings on the corresponding wires. The value of one revolution of the screw is, p. 53, $R = 45''.042 = 3^s.003$.

I	II	III	IV	V
29.966	27.443	24.891	22.361	19.797
30.130	27.620	25.054	22.523	19.965
29.969	27.445	24.890	22.357	19.799
30.131	27.621	25.058	22.527	19.970
29.968	27.448	24.895	22.363	19.802
30.135	27.618	25.060	22.528	19.969
29.968	27.445	24.898	22.356	19.797
30.135	27.623	25.062	22.526	19.969
30.050	27.533	24.976	22.443	19.883

$$i_1 = (30.050 - 24.976)\,R = +15^s.237;$$
$$i_2 = (27.533 - 24.976)\,R = +\ \ 7\,.679;$$
$$i_4 = (22.443 - 24.976)\,R = -\ \ 7\,.607;$$
$$i_5 = (19.883 - 24.976)\,R = -\ 15.294.$$

* More accurate readings will be obtained, as in many other cases, by setting the micrometer wire on each side of the fixed wire and just in contact with it. The mean of the readings in the two positions is the reading for the coincidence of the two wires.

$$i_m = +0^s.003 \text{ for clamp west,}$$
$$i_m = -0\ .003 \text{ for clamp east.}$$

(b). Observe the transits of a close circumpolar star over the several wires, and solve (203) for the intervals i. It is convenient and sufficiently accurate to use (203) in the form

$$i = \sin I \frac{\cos \delta}{15 \sin 1''}. \tag{213}$$

Solving as in the case of (137), the resulting values of i are expressed in seconds of time.

Example. 1891 March 16, λ *Ursae Minoris* was observed at lower culmination with the transit instrument of the Detroit Observatory, clamp east, as below. Required the wire intervals. From the Naut. Alm., p. 304, $\delta = 88°\ 57'\ 47''.3$.

Wires.	Chronom.	I	I	$\sin I$	i
I	$7^h\ 2^m\ 37^s$	$-14^m\ 3^s$	$-3°\ 30'\ 45''$	8.787222_n	$-15^s.245$
II	9 35	$-\ 7\ 5$	$-1\ 46\ 15$	8.489986_n	$-\ 7\ .689$
III	16 40				
IV	23 40	$+\ 7\ 0$	$+1\ 45\ 0$	8.484848	$+\ 7\ .599$
V	30 46	$+14\ 6$	$+3\ 31\ 30$	8.788762	$+15\ .299$

A number of stars should be observed in this way, and the mean of all the results adopted as the wire intervals.

DETERMINATION OF THE LEVEL CONSTANT.

98. The level constant b is generally found by means of a spirit level, as explained in §§ 61 and 62. However, the level is applied to the upper surface of the cylindrical pivots and does not give the inclination of the axis, which passes through their centres, unless their radii are equal.

Fig. 15.

To determine *the inequality of the pivots* and the method of eliminating it, let A and B, Fig. 15, be the centres of the west and east pivots, clamp west; M and M' the vertices of the V's in which the pivots rest; L and L' the vertices of the V's of the

level; and HM' a horizontal line. Then $BAC = BAD$ is the inequality of the pivots, which we shall represent by p. If we let

$B' =$ the inclination given by the level for clamp west,
$B'' =$ " " " " " " " " east,
$b' =$ " true inclination for clamp west,
$b'' =$ " " " " " east,
$\beta =$ the constant angle $HM'M$,

we can write

$$b' = B' + p = \beta - p, \text{ for clamp west,} \tag{214}$$
$$b'' = B'' - p = \beta + p, \text{ for clamp east;} \tag{215}$$

and therefore

$$p = \frac{B'' - B'}{4}. \tag{216}$$

The value of p should be determined a large number of times, and the mean of all the individual results adopted as its final value. In making the observations the telescope should be set at different zenith distances, to detect any variations from a cylindrical form of the pivots.

Example. 1891 Feb. 17. The following observation was made on the pivots of the Detroit Observatory transit instrument. Required the inequality of the pivots and the inclinations of the rotation axis. The value of one division of the striding level is $d = 1''.878 = 0^s.125$. [See § 64].

Clamp.	Zenith distance.	Level direct.		Level reversed.	
		w	e	w'	e'
W.	N. 30°	7.2	9.6	15.6	1.3
E.	S. 30	5.2	11.7	13.5	3.4

From (144),

$$b = B' = +2.975\,d = +0^s.372, \text{ for clamp west;}$$
$$b = B'' = +0.900\,d = +0^s.112, \text{ for clamp east.}$$

Substituting these values in (216), we find

$$p = -0^s.065;$$

and therefore, from (214) and (215),

$$b' = +0^s.372 - 0^s.065 = +0^s.307,$$
$$b'' = +0^s.112 + 0^s.065 = +0^s.177.$$

The mean of twenty-two determinations of p for this instrument gave $p = -0^s.066 \pm 0^s.001$.

See also § 100, (c), for another method.

99. We have supposed that the V's in which the pivots rest and the V's of the level are equal, as is usually the case. If they are unequal, let

$2i$ = the angle of the level V,
$2i_1$ = the angle of the V of the pivot bearing;

and it can be shown that the inequality of the pivots is given by

$$p = \frac{B'' - B'}{2} \cdot \frac{\sin i_1}{\sin i + \sin i_1}. \tag{217}$$

DETERMINATION OF THE COLLIMATION CONSTANT.

100. (a). *By a distant terrestrial object.* Place the telescope in a horizontal position and select some well defined distant point whose image is seen near the middle wire. With the micrometer measure the distance of the image from the middle wire. For clamp west call this distance D', and let it be considered positive if the image is west of the middle wire, negative if east. Reverse the axis and measure this distance again, calling it D'' for clamp east; D'' being positive or negative if the image is west or east of the middle wire. The collimation constant is given by

$$c = \frac{D'' - D'}{2}, \text{ for clamp west.} \tag{218}$$

For clamp east the sign of c is reversed.

Example. 1891 April 4. The following observations on a distant object nearly in the horizon were made with the transit instrument of the Detroit Observatory. Required the value of c. The value of one revolution of the screw is $3^s.003$.

Clamp.	Image.	Micrometer on image.	Micrometer on III
W.	W. of III	34.067	24.792
		.065	.795
		.058	.793
		Mean 34.063	Mean 24.793
E.	W. of III	15.750	24.794
		.744	.792
		.752	.795
		Mean 15.749	Mean 24.794

We have

$$D' = (34.063 - 24.793)\,R = +\,9.270\,R,$$
$$D'' = (24.794 - 15.749)\,R = +\,9.045\,R;$$

and therefore, from (218),

$$c = -\,0.112\,R = -\,0^s.336, \text{ for clamp west.}$$

(*b*). *By a circumpolar star.* Observe the transit of a close circumpolar star over the first two or three wires; then quickly reverse the instrument and observe the transit over as many of the same wires as possible, being sure to determine the level constant both before and after reversing. Reduce the times of transit in the two positions to the equivalent times of crossing the middle wire. Let θ_1 and θ_2 be these times, and let b' and b'' be the level constants for clamp west and clamp east. Then by (211), for clamp west,

$$\Delta\theta = a - \theta_1 - aA - b'B - cC + 0^s.021 \cos\phi\, C;^*$$

and for clamp east

$$\Delta\theta = a - \theta_2 - aA - b''B + cC + 0^s.021 \cos\phi\, C.$$

Subtracting and solving for c we obtain

$$c = \tfrac{1}{2}(\theta_2 - \theta_1)\cos\delta + \tfrac{1}{2}(b'' - b')\cos(\phi - \delta). \tag{219}$$

For lower culmination δ is replaced by $180^\circ - \delta$, as before, so that

$$c = -\,\tfrac{1}{2}(\theta_2 - \theta_1)\cos\delta - \tfrac{1}{2}(b'' - b')\cos(\phi + \delta). \tag{220}$$

An example is given in § 105.

(*c*). *By the nadir.* If the telescope be directed vertically downward to a basin of mercury, and a piece of glass be placed diagonally over the eye piece in such a way that the light from a lamp at one side will be thrown on the reticule, the middle wire and its image reflected from the mercury may be seen near together. Measure with the micrometer the distance between the middle wire and its reflected image. Let M be this distance, and consider it positive when the wire is west of its image. If the rotation axis is horizontal we have $M = 2c$; but if there is a level constant b, the distance is diminished by $2b$, so that $M = 2c - 2b$; or

$$c = \tfrac{1}{2}M + b. \tag{221}$$

The sign of c corresponds to the position of the clamp at that time.

If we wish to determine the level and collimation constants

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* The correction for rate will be small compared with the probable error of a transit of a slowly-moving northern star, and may be neglected.

at the same time, measure the distance that the middle wire is west of its image in the two positions of the clamp. Let

M' = the distance for clamp west,
M'' = " " " " east,
b' = the level constant for clamp west,
b'' = " " " " " east.

We have, for clamp west,

$$c = \tfrac{1}{2} M' + b',$$

and for clamp east,

$$-c = \tfrac{1}{2} M'' + b''.$$

Therefore

$$c = \tfrac{1}{4}(M' - M'') + \tfrac{1}{2}(b' - b''),$$
$$b' + b'' = -\tfrac{1}{2}(M' + M'').$$

From (214) and (215), $b' - b'' = -2p$. Therefore

$$c = \tfrac{1}{4}(M' - M'') - p, \text{ clamp west,} \tag{222}$$
$$c = -\tfrac{1}{4}(M' - M'') + p, \text{ clamp east,} \tag{223}$$
$$b' = -\tfrac{1}{4}(M' + M'') - p, \text{ clamp west,} \tag{224}$$
$$b'' = -\tfrac{1}{4}(M' + M'') + p, \text{ clamp east.} \tag{225}$$

Example. 1891 July 24. The following nadir observations were made with the transit instrument of the Lick Observatory. Required the values of c, b' and b''.

Clamp.	Micrometer on middle wire.	Micrometer on image.
W.	11.025	10.847
	.125	10.852
	Mean 11.075	Mean 10.849
E.	11.020	11.164
	.135	.166
	Mean 11.077	Mean 11.165

The middle wire was east of its image.

For this instrument, $p = -0^s.021$, $R = 2^s.931$. We have

$$M' = -(11.075 - 10.849)\,R = -0^s.662,$$
$$M'' = -(11.165 - 11.077)\,R = -0.258.$$

Therefore

$$c = -0^s.101 + 0^s.021 = -0^s.080, \text{ clamp west,}$$
$$b' = +0.230 + 0.021 = +0.251,$$
$$b'' = +0.230 - 0.021 = +0.209.$$

DETERMINATION OF THE AZIMUTH CONSTANT.

101. The azimuth constant a can be determined only from observations of stars. Let two stars (α_1, δ_1) and (α_2, δ_2) be observed. Since all the constants except a are now determined, the times of observation of the two stars can be corrected for all

errors save the azimuth. Let θ_1 and θ_2 be the times so corrected. Then (211) reduces for the first star, to

$$\Delta\theta = a_1 - \theta_1 - aA_1,$$

and for the second star, to

$$\Delta\theta = a_2 - \theta_2 - aA_2,$$

A_1 and A_2 being the values of A corresponding to δ_1 and δ_2. Combining these equations, we obtain

$$a = \frac{(a_1 - \theta_1) - (a_2 - \theta_2)}{A_1 - A_2}. \tag{226}$$

It will be seen that to determine a accurately all the other constants of the instrument must be well determined, since errors in any one or more of them affect the values of θ_1 and θ_2. If the instrument is not mounted in a very stable manner, the right ascensions α_1 and α_2 should differ as little as possible. The value of a will be determined best when the denominator $A_1 - A_2$ is as large as possible. If both stars are observed at upper culmination, one must be as far south as possible and the other as near the pole as possible, in which case A_1 and A_2 will be large and opposite in sign. This condition will be fulfilled still better by observing one star (α_1, δ_1) at lower culmination and the other (α_2, δ_2) at upper culmination, both as near the pole as possible and differing nearly 12^h in right ascension. In this case α_1 must be replaced by $12^h + \alpha_1$ and δ_1 by $180° - \delta_1$ in the various formulæ. Stars observed at lower culmination are marked *S. P.*, (*sub polo*).

102. When the chronometer correction is required to be known very accurately, it is customary to observe the transits of ten or twelve stars. The observing list should be made out very carefully in advance. Half the stars should be observed with clamp west, the other half with clamp east, since any errors in the adopted values of i_m, p and c will be practically eliminated by reversing the axis. To determine a well, a pair of azimuth stars should be observed before reversing, and another pair after reversing. The remaining stars on the list should be those which culminate near the zenith, or between the zenith and equator; since the zenith stars are affected least by an error in the adopted value of a, and the time of transit can be estimated most accurately for the rapidly moving equatorial stars. There is no method of eliminating an error in b, and it must be very carefully determined. A good program to follow is—

Take the level readings.
Observe half the stars.
Take the level readings.
Reverse the instrument.
Take the level readings.
Observe half the stars.
Take the level readings.

If there is time between the stars for making further level readings they should be made. In reversing, the instrument must be handled very carefully to avoid changing the constants.

ADJUSTMENTS.

103. To set up the instrument, it should first be placed by estimation as nearly as possible in the meridian, and the follow-adjustments made:

1st. To bring the wires in the common focus of the eye piece and objective, slide the eye piece in or out until the wires are perfectly well defined; then direct the telescope to a very distant terrestrial object, or to a star, and move the tube carrying the wires and eye piece until an image of the object seen on one of the wires will remain on the wire when the position of the eye is changed.

2d. Make the level constant very nearly zero, testing it by the method of § 98.

3d. To make the wires perpendicular to the axis, direct the telescope to a well defined mark and bisect it with the middle wire. Adjust the reticule so that the object remains on the wire when the telescope is moved up and down.

4th. Test the collimation by the method of § 100, (a), and move the reticule sidewise until c is made very small.

5th. To set the finding circle, direct the telescope to a bright star near the zenith, whose declination is δ. When the star enters the field of view move the telescope so that the star describes a diameter of the field, (usually marked by one or two wires), and clamp the instrument. If the circle is designed to give the zenith distances, set it at the reading

$$z = \phi - \delta.$$

It will then read correctly for all other stars.

6th. To adjust it in azimuth, direct the telescope to a star near the zenith whose right ascension is α_1. Observe its transit over the middle wire, and let the chronometer time of transit be θ_1. The approximate chronometer correction is

$$\Delta\theta_1 = \alpha_1 - \theta_1.$$

Set the telescope for a circumpolar star whose right ascension is α_2. It culminates at the chronometer time

$$\theta_2 = \alpha_2 - \Delta\theta_1 .$$

Rotate the whole instrument horizontally so that the star is on the middle wire at the instant when the chronometer indicates the time θ_2.

7th. Repeat the 2d adjustment.

8th. Repeat the 6th adjustment.

9th. The final adjustment in azimuth should be tested by the method of § 101.

104. *Example.* Wednesday, 1891 Feb. 25. The following observing list was prepared and the stars observed by the eye and ear method on the transit instrument of the Detroit Observatory, to determine the correction to sidereal chronometer Negus no. 721. The stars were selected from the list in the *Berliner Astronomisches Jahrbuch*, pp. 190-327. For convenience in referring to them in the reductions they are numbered, together with the level observations, in the first column. Their magnitudes are given in the third column. The "Setting" is the reading at which the circle is to be set for observing each star. The circle of this instrument reads zero when the telescope points to the zenith and the degrees are numbered both directions from the zero. The setting is therefore the zenith distance.

No.	Object.	Mag.	α	δ	Setting.
(1)	Level				
(2)	*π Cephei, S. P.*	4.6	$23^h\ 4^m\ 20^s$	74° 47′.9	N. 62° 55′
(3)	*δ Leonis*	2.3	11 8 20	21 7	S. 21 10
(4)	*ν Ursae Majoris*	3.3	12 37	33 41	S. 8 36
(5)	*σ Leonis*	4.1	15 32	6 38	S. 35 39
(6)	*λ Draconis*	3.3	25 0	69 56	N. 27 39
(7)	Level Reverse				
(8)	Level				
(9)	*χ Ursae Majoris*	3.8	40 19	48 23	N. 6 6
(10)	*β Leonis*	2.0	43 31	15 11	S. 27 6
(11)	*β Virginis*	3.3	45 2	2 23	S. 39 54
(12)	*γ Ursae Majoris*	2.3	48 8	54 18	N. 12 1
(13)	Level				
(14)	*ε Corvi*	3.0	12 4 32	−22 1	S. 64 18
(15)	*4 H. Draconis*	4.6	7 12	78 13 .2	N. 35 56
(16)	Level				

The level observations and their reductions are

(1)		(7)		(8)		(13)		(16)	
W.	E.	W.	E.	W.	E.	W.	E.	W.	E.
15.1	9.1	15.1	9.3	15.3	9.0	14.3	10.3	14.6	10.1
12.0	12.11	3.7	10.8	8.6	15.9	11.0	13.5	11.0	13.5
12.1	12.1	13.6	10.8	9.0	15.4	10.7	13.9	11.1	13.5
15.2	8.9	14.8	9.5	15.1	9.4	14.0	10.5	14.4	10.1

	(1)	(7)		(8)	(13)	(16)
$B' =$	$+1\ .525d$	$+2\ .100d$	$B'' =$	$-0\ .212d$	$+0\ .225d$	$+0\ .487d$
$B' =$	$+0^s.191$	$+0^s.262$	$B'' =$	$-0^s.026$	$+0^s.028$	$+0^s.061$
$p =$	$-0^s.066$	$-0^s.066$	$p =$	$-0^s.066$	$-0^s.066$	$-0^s.066$
$b' =$	$+0^s.125$	$+0^s.196$	$b'' =$	$+0^s.040$	$+0^s.094$	$+0^s.127$

The times of transit over the five wires are given below. For clamp east, and for lower culmination clamp west, the transits occurred in the order V, IV, III, II, I. The mean of the observed times is given in the column θ_m. The values of b are those found by interpolating for the instant of observation, assuming its value to vary uniformly with the time between two consecutive determinations.

Cl.	Object.	I	II	III	IV	V	θ_m	b
W.	(1)	s	s	s	s	h m s	h m s	$+0^s.125$
"	(2)	43.9	14.6	45.3	16.0	10 48 47.4	10 49 45.44	.136
"	(3)	26.8	35.0	43.0	51.1	53 59.3	53 43.04	.146
"	(4)	41.8	50.9	0.0	9.2	58 18.3	58 0.04	.156
"	(5)	40.0	47.6	55.3	3.0	11 1 10.8	11 0 55.34	.164
"	(6)	37.1	59.3	21.5	43.9	11 6.1	10 21.60	.188
"	(7)							.196
E.	(8)							.040
"	(9)	5.6	54.3	42.9	31.2	25 19.6	25 42.72	.057
"	(10)	10.2	2.4	54.5	46.5	28 38.6	28 54.44	.067
"	(11)	40.7	33.2	25.6	17.9	30 ——	30 29.35	.072
"	(12)	57.2	44.2	31.1	18.0	33 4.8	33 31.06	.080
"	(13)							.094
"	(14)	12.1	3.9	55.6	47.4	49 39.3	49 55.66	.114
"	(15)	49.2	11.5	34.2	56.4	51 19.4	52 34.14	.120
"	(16)							.127

Observations for determining the collimation constant c were made by the method (a), § 100, on the preceding afternoon and the following forenoon, which gave for clamp west $c = +0^s.112$ and $c = +0^s.108$, respectively. We shall adopt their mean, $c = +0^s.110$.

The hourly rate of the chronometer was $+0^s.15$. Let it be required to determine the chronometer correction at the time $\theta_o = 11^h\ 20^m$. The correction for rate is $(\theta_m - 11^h\ 20^m)0^s.15$.

For convenience, let $c - 0^s.021 \cos\varphi + i_m = c'$, and we have

$$c' = +0^s.110 - 0^s.015 + 0^s.003 = +0^s.098, \text{ for clamp west,}$$
$$c' = -0^s.110 - 0^s.015 - 0^s.003 = -0^s.128, \text{ for clamp east.}$$

Star (11) was observed over only the first four wires. In this case $i_m = -\frac{1}{4}(i_1 + i_2 + i_4) = -3^s.825$, and $c' = -0^s.110 - 0^s.015 - 3^s.825 = -3^s.950$.

The values of A, B and C are taken from a table computed for the latitude of the Detroit Observatory. They are used here to three decimal places; for ordinary work two are sufficient. To illustrate the application of (200) and (212), we shall compute A, B and C for the first two stars.

	(2)		(2)		(3)		(3)
δ	$74^\circ\ 47'.9$	A	$+3.395$	δ	$21^\circ\ 7'$	A	$+0.387$
ϕ	$42\ \ 16\ .8$	B	-1.736	ϕ	$42\ \ 17$	B	$+1.000$
$\sin(\phi + \delta)$	9.9496	C	-3.813	$\sin(\phi - \delta)$	9.5576	C	$+1.072$
$\cos(\phi + \delta)$	9.6582_n			$\cos(\phi - \delta)$	9.9697		
$\sec\delta$	0.5813			$\sec\delta$	0.0302		

The apparent right ascensions are taken as accurately possible from the *Jahrbuch*. We are now prepared to fill in the columns A, B, C, *Rate*, Cc', Bb, and α; afte which we can determine a, and thence Aa and θ'.

Star.	A	B	C	*Rate.*	Cc'	Bb	Aa	θ'	α	$\Delta\theta$	Wt.
				s	s	s	s	h m s	h m s	m s	
(2)	+3.395	−1.736	−3.813	−.08	−0.37	−.24	−1.36	10 49 43.39	11 4 20.01	+14 36.62	0
(3)	+ .387	1.000	1.072	−.07	+ .11	+.15	− .15	53 43.08	8 19.66	36.58	2
(4)	+ .179	+1.187	+1.200	−.06	+ .12	+.19	− .07	58 0.22	12 36.77	36.55	2
(5)	+ .587	+ .818	+1.007	−.05	+ .10	+.13	− .23	11 0 55.29	15 31.78	36.49	2
(6)	−1.353	+2.582	+2.915	−.02	+ .29	+.49	+ .54	10 22.90	24 59.52	36.62	1
(9)	− .160	1.497	+1.506	−.01	− .19	+.09	.06	25 42.69	40 19.26	36.57	2
(10)	+ .472	+ .922	+1.036	+.02	− .13	+.06	− .16	28 54.23	43 30.82	36.59	2
(11)	+ .642	+ .768	+1.001	+.03	−3.95	+.06	− .22	30 25.27	45 1.77	36.50	2
(12)	− .357	+1.676	+1.714	+.03	− .22	+.13	+ .12	33 31.12	48 7.67	36.55	1
(14)	+ .972	+ .468	+1.078	+.07	− .14	+.05	− .34	49 55.30	12 4 31.81	36.51	1
(15)	−2.875	+3.966	+4.898	+.08	− .63	+.48	+1.00	52 35.07	7 11.58	+14 36.51	0

Using stars (2) and (6) to determine a we have

$$\begin{aligned} \alpha_1 &= 11^h\ 4^m\ 20^s.01, & \alpha_2 &= 11^h\ 24^m\ 59^s.52,\\ \theta_1 &= 10\ \ 49\ \ 44\ .75, & \theta_2 &= 11\ \ 10\ \ 22\ .36,\\ \alpha_1 - \theta_1 &= +\ 14\ \ 35\ .26, & \alpha_2 - \theta_2 &= +\ 14\ \ 37\ .16,\\ A_1 &= +3.395, & A_2 &= -1.353, \end{aligned}$$

and therefore $a = -0^s.400$. Similarly, from (14) and (15) we obtain $a = -0^s.348$. Using these as the values of a for clamp west and east respectively, we form the column Aa. All the corrections have now been computed. Substituting them in (211) for each star, we obtain the values $\Delta\theta$.

Stars (2) and (15) were observed solely to determine a, and

the values of $\Delta\theta$ furnished by them will be given a weight 0, in the last column. Assigning a weight 2 to those which culminate near the zenith and between the zenith and equator, and a weight 1 to those outside those limits, for the reasons given in § 102, we obtain for the weighted mean of the chronometer corrections,

$$\Delta\theta = + 14^m\ 36^s.55 \pm 0^s.009,$$

which we shall adopt as the chronometer correction at the time $\theta_0 = 11^h\ 20^m$.

105. To illustrate the determination of c by the method of § 100, (b), *Polaris* was observed at lower culmination the same night, as below.

Polaris.			Level.			
			Clamp W.		Clamp E.	
			W.	E.	W.	E.
Clamp W.	V	$12^h\ 52^m\ 6^s$	13.9	10.2	9.9	14.0
" "	IV	12 57 51	12.0	12.0	11.4	12.5
Reversed.			12.1	12.0	11.4	12.4
Clamp E.	III	13 3 22	13.8	10.2	9.8	14.0
" "	IV	13 9 3				
" "	V	13 14 54				

The intervals of time required for *Polaris* to pass from V to III and from IV to III are given by (203,) first putting it in the form

$$\sin I = 15 \sin 1'' \sec \delta .\ i. \tag{227}$$

The value of δ was $+ 88°\ 43'\ 49''$. Substituting $i_4 = 7^s.607$ and $i_5 = 15^s.294$ successively for i in the formula, we find

$$I_4 = 1°\ 25'\ 50'' = 5^m\ 43^s.3 \qquad I_5 = 2°\ 52'\ 37'' = 11^m\ 30^s.5;$$

and therefore the equivalent times of transit over III are

Clamp W.	Clamp E.
$12^h\ 52^m\ 6^s + 11^m\ 30^s.5 = 13^h\ 3^m\ 36^s.5$	$13^h\ 3^m\ 22^s = 13^h\ 3^m\ 22^s.$
$12\ 57\ 51 + 5\ 43 .3 = 13\ 3\ 34 .3$	$13\ 9\ 3 - 5^m\ 43^s.3 = 13\ 3\ 19 .7$
	$13\ 14\ 54 - 11\ 30 .5 = 13\ 3\ 23 .5$

Taking the means for clamp west and clamp east we obtain

$$\theta_1 = 13^h\ 3^m\ 35^s.4, \qquad \theta_2 = 13^h\ 3^m\ 21^s.7.$$

The level constants given by the above observations are

$$b' = + 0^s.050, \qquad b'' = - 0^s.096.$$

Substituting these in (220) we obtain

$$c = + 0^s.091, \text{ clamp west.}$$

REDUCTION BY THE METHOD OF LEAST SQUARES.

106. In case the chronometer correction is required with all possible accuracy, the series of transit observations should be reduced by the method of least squares. Let us assume that the level constant, the rate and i_m are accurately determined, and that the chronometer correction, the azimuth constant and the collimation constant are to be obtained from the observations. To avoid dealing with large quantities, let $\Delta\theta_0$ be an approximate value of $\Delta\theta$, and x a small correction to $\Delta\theta$, such that

$$\Delta\theta_0 + x = \Delta\theta. \tag{228}$$

Further, let

$$d = \Delta\theta_0 + \theta_m + Bb - 0^s.021 \cos\phi\, C + i_m\, C + (\theta_m - \theta_0)\, \delta\theta - \alpha; \tag{229}$$

and (211) takes the form

$$Aa \pm Cc + x + d = 0, \tag{230}$$

the lower sign being for clamp east. A value for $\Delta\theta_0$ having been assumed, all the terms in the second member of (229) are known for each star. Therefore, a, c and x are the only unknown quantities in (230). Each star furnishes an equation of this form, and the solution by the method of least squares gives their most probable values; and therefore, by (228), the most probable value of $\Delta\theta$.

The time of transit of a star over a wire can be estimated more accurately for a rapidly moving equatorial star than for any other. Assistant Schott of the Coast Survey* investigated the subject by discussing a large number of observations, and found that the probable error of the observed time of transit over one wire is best represented by

$$\varepsilon = \sqrt{(0.063)^2 + (0.036)^2 \tan^2\delta}, \text{ for large instruments;}$$
$$\varepsilon = \sqrt{(0.080)^2 + (0.063)^2 \tan^2\delta}, \text{ for small instruments.}$$

The values of ε given in the table below are computed from these for the different values of δ. If 1 be the weight of an observation of an equatorial star, ε_0 its probable error, and p the weight of an observation of any other star we have, from theory,

$$p = \frac{\varepsilon_0^2}{\varepsilon^2}.$$

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* See U. S. Coast and Geodetic Survey Report for 1880.

For large instruments, $\varepsilon_0 = 0^s.063$, and for small ones, $\varepsilon_0 = 0^s.080$. Substituting the values of ε in this equation we find the following values of p.

δ	Large instruments.			Small instruments.		
	ε	p	$\sqrt{p}$	ε	p	$\sqrt{p}$
0°	$\pm 0^s.06$	1.00	1.00	$\pm 0^s.08$	1.00	1.00
10	.06	1.00	1.00	.08	.98	1.00
20	.06	.98	1.00	.08	.92	.96
30	.07	.91	.95	.09	.83	.91
40	.07	.82	.90	.10	.70	.83
50	.08	.69	.83	.11	.53	.73
55	.08	.61	.78	.12	.44	.66
60	.09	.51	.71	.14	.34	.59
65	.10	.40	.63	.16	.26	.51
70	.12	.29	.54	.19	.18	.42
75	.15	.18	.43	.25	.10	.32
80	.21	.09	.30	.37	.05	.22
85	.42	.02	.15	.72	.01	.11
86	.52	.015	.122	.90	.008	.088
87	.69	.008	.091	1.21	.004	.066
88	1.03	.004	.061	1.82	.002	.044
89	2.06	.001	.031	3.70	.000	.022
90	∞	.000	.000	∞	.000	.000

The observation equations (230) should be multiplied through by the square roots of their respective weights before forming the normal equations. (230) becomes

$$\sqrt{p}\,(Aa \pm Cc + x + d) = 0. \qquad (231)$$

107. In case one or more wires have been missed, the weight is diminished. If we let

N = the whole number of wires,
n = the number of wires observed,
1 = the factor for an observation over the N wires,
P = the factor for an observation over n wires;

the weight for an incomplete transit is pP.

Assistant Schott found that we should use

$$P = \frac{1 + \frac{1.6}{N}}{1 + \frac{1.6}{n}}, \text{ for large instruments,} \qquad (232)$$

$$P = \frac{1 + \frac{2.0}{N}}{1 + \frac{2.0}{n}}, \text{ for small instruments.} \tag{233}$$

The following table gives the value of P for reticules containing seven and five wires, for the different values of n.

Large instruments.

n	P	n	P
7	1.00	5	1.00
6	.97	4	.94
5	.93	3	.86
4	.88	2	.73
3	.80	1	.51
2	.68		
1	.47		

Small instruments.

n	P	n	P
7	1.00	5	1.00
6	.96	4	.93
5	.92	3	.84
4	.86	2	.70
3	.77	1	.47
2	.64		
1	.43		

108. We shall now apply these methods to the reduction of the transit observations in § 104.

We shall assume $\Delta\theta_0 = +14^m\,36^s.5$. The values of θ_m, Bb, $(\theta_m - \theta_0)\,\delta\theta$, and α are obtained as before and we shall use their values tabulated in § 104. To compute the terms $-0^s.021 \cos\varphi\, C$ and $i_m\, C$, let $-0^s.021 \cos\varphi + i_m = c''$. Thus

$$c'' = -0^s.015 + 0^s.003 = -0^s.012, \text{ for clamp west,}$$
$$c'' = -0\,.015 - 0\,.003 = -0\,.018, \text{ for clamp east,}$$
$$c'' = -0\,.015 - 3\,.825 = -3\,.840, \text{ for star (11).}$$

The products Cc'' are given in the table below. The value of d is found for each star by (229). The column $\sqrt{p}$ is taken from the table for the large instruments; but for star (11), which is incomplete, the weight is found from pP.

Star.	Cc''	d	$\sqrt{p}$
(2)	$+0^s.05$	$+1^s.66$	0.43
(3)	$-\ \ .01$	$-\ \ .05$	.99
(4)	$-\ \ .01$	$-\ \ .11$	.93
(5)	$-\ \ .01$	$+\ \ .13$	1.00
(6)	$-\ \ .03$	$-\ \ .98$	.54
(9)	$-\ \ .03$	$+\ \ .03$	.84
(10)	$-\ \ .02$	$+\ \ .18$	1.00
(11)	$-3\ .84$	$+\ \ .33$	.97
(12)	$-\ \ .03$	$+\ \ .02$	.79
(14)	$-\ \ .02$	$+\ \ .45$	.99
(15)	$-\ \ .09$	$-\ \ .47$	.34

Substituting the values of A, C, d and $\sqrt{p}$ in (231), being careful to change the sign of the coefficient of c for clamp east, we have the weighted observation equations

$$\left.\begin{array}{rrrr} +1.462a & -1.640c & +0.43x & +0.714 = 0, \\ +\ .383 & +1.061 & +\ .99 & -\ .049 = 0, \\ +\ .166 & +1.116 & +\ .93 & -\ .102 = 0, \\ +\ .587 & +1.007 & +1.00 & +\ .130 = 0, \\ -\ .731 & +1.574 & +\ .54 & -\ .529 = 0, \\ -\ .134 & -1.267 & +\ .84 & +\ .025 = 0, \\ +\ .472 & -1.036 & +1.00 & +\ .180 = 0, \\ +\ .623 & -\ .971 & +\ .97 & +\ .320 = 0, \\ -\ .282 & -1.354 & +\ .79 & +\ .016 = 0, \\ +\ .962 & -1.067 & +\ .99 & +\ .445 = 0, \\ -\ .977 & -1.696 & +\ .34 & -\ .160 = 0. \end{array}\right\} \tag{234}$$

The normal equations formed from these are

$$\left.\begin{array}{rrrr} +5.780a & -\ 2.278c & +2.714x & +2.331 = 0, \\ -2.278 & +18.020 & -2.505 & -2.807 = 0, \\ +2.714 & -\ 2.505 & +7.689 & +0.918 = 0. \end{array}\right\} \tag{235}$$

Their solution gives

$$a = -0^s.383, \qquad c = +0^s.115, \qquad x = +0^s.053;$$

and therefore

$$\Delta\theta = \Delta\theta_0 + x = +14^m\,36^s.5 + 0^s.053 = +14^m\,36^s.553.$$

The weights of the quantities just determined are

$$p\,(a) = 4.71, \qquad p\,(c) = 16.80, \qquad p\,(x) = 6.29.$$

Substituting the values of a, c and x in (234), we obtain the the residuals $\sqrt{p}\ v$,

$$-0.012, -.021, +.011, +.074, -.039. -.024, -.067, +.021, +.010, +.007, +.001,$$

The sum of the squares of these is $\Sigma\, p\, vv = 0.0134$. The probable error r_1 of an observation of weight unity is given by

$$r_1 = \pm\, 0.674 \sqrt{\frac{\Sigma\, p\, vv}{m - q}}, \tag{236}$$

where m is the number of observation equations, and q is the number of unknown quantities. In this case $m = 11$ and $q = 3$. Therefore $r_1 = \pm\, 0^s.028$.

The probable errors of the unknowns are given by

$$r\,(a) = \frac{r_1}{\sqrt{p\,(a)}}, \quad r\,(c) = \frac{r_1}{\sqrt{p\,(c)}}, \quad r\,(x) = \frac{r_1}{\sqrt{p\,(x)}}. \tag{237}$$

Therefore

$$r\,(a) = \pm\, 0^s.013, \qquad r\,(c) = \pm\, 0^s.007, \qquad r\,(x) = \pm\, 0^s.011.$$

and

$$a = -0^s.383 \pm 0^s.013,$$
$$c = +0^s.115 \pm 0^s.007,$$
$$\Delta\theta = +14^m\,36^s.553 \pm 0^s.011.$$

CORRECTION FOR FLEXURE.

109. In the broken or prismatic transit instrument, (§ 94), a correction for flexure due to the bending of the axis must be applied. The effect of the flexure is to lower the reflecting prism without changing the position of the eye piece, which is the same as changing the inclination of the axis. It can therefore be allowed for by changing the measured inclination b, using

$b + f$ for clamp west,
$b - f$ for clamp east,

f being the coefficient of flexure, and the eye piece being on the clamp end of the axis.

It requires special apparatus to determine f directly, so that unless its value for a particular instrument has been well determined, it is best to reduce all the transit observations by the method of least squares, inserting another unknown quantity f, thus:

$$\sqrt{p}\,(Aa \pm Bf \pm Cc + x + d = 0. \qquad (238)$$

GEOGRAPHICAL LONGITUDE.

110. The accurate determination of the difference of longitude of two places requires the accurate determination of the time at each place and a method of comparing these times. One of the three following methods of comparison is generally employed.

(*a*). *By Transportation of Chronometers.* Let the eastern place be E, the western place W, and the difference of their longitude L. Determine the correction $\Delta\theta_e$ and the rate $\delta\theta$ of a chronometer at E at the chronometer time θ_e. Carry the chronometer to W and there determine its correction $\Delta\theta_w$ at the chronometer time θ_w. Then

$\theta_w + \Delta\theta_w =$ correct time at W at chronometer time θ_w;
$\theta_w + \Delta\theta_e + \delta\theta\,(\theta_w - \theta_e) =$ " " " E " " " θ_w.

Their difference is

$$L = \Delta\theta_e + \delta\theta\,(\theta_w - \theta_e) - \Delta\theta_w. \qquad (239)$$

The rate of the chronometer during transportation differs

from its rate when at rest. The change may be eliminated largely by transporting it in both directions between E and W. The rate is also a function of the temperature and the lubrication of the pivots. It has been found that the rate m at any temperature ϑ can be represented by the formula

$$m = m_0 + k(\vartheta - \vartheta_0) - k't, \tag{240}$$

in which ϑ_0 is the temperature of best compensation, m_0 the rate at that temperature with $t = 0$, t the time measured from that instant, k the temperature coefficient and k' the lubrication coefficient. By determining m_0, k, ϑ_0 and k' for each chronometer, keeping a record of the temperature during transportation, and transporting several chronometers in both directions, the method yields good results. It should never be employed, however, except when the telegraphic method is impracticable.

(b). *By the Electric Telegraph.* This is by far the most accurate and convenient method known. The observers at E and W first determine their chronometer corrections by transit observations. Next, the observer at E taps the signal key of the telegraph line joining E and W simultaneously with the beats of his chronometer, and the observer at W notes on his chronometer the times of receiving these signals. In the same way the observer at W sends return signals to the observer at E. The chronometer corrections are now redetermined by transit observations. Let

θ_e = correct time at E of sending signal,
θ_w = " " " W " receiving signal,
θ_w' = " " " W " sending return signal,
θ_e' = " " " E " receiving return signal,
μ = the transmission time.

Then

$$\theta_e + \mu - L = \theta_w,$$
$$\theta_e' - \mu - L = \theta_w'.$$

Therefore

$$L = \tfrac{1}{2}(\theta_e + \theta_e') - \tfrac{1}{2}(\theta_w + \theta_w'), \tag{241}$$
$$\mu = \tfrac{1}{2}(\theta_w - \theta_w') - \tfrac{1}{2}(\theta_e - \theta_e'). \tag{242}$$

There are several small errors affecting the value of L obtained by this method, viz.:

1st. The personal errors of the observers in sending and receiv- the signals;

2d. The time required to close the circuit after the finger touches the key, and to move the armature of the receiving magnet through the space in which it plays and give the click;

3d. The personal errors of the observers in getting the chronometer corrections, and errors in the right ascensions of the stars employed.

These must be eliminated as far as possible in refined determinations, which is best done by a modification of the above method, called the *method of star signals*. *One* clock or chronometer is placed in the circuit of the telegraph line, and at each station a chronograph and the signal key of a transit instrument are placed in the same circuit. The same list of stars is observed at both places, thereby eliminating errors in the right ascensions. When the first star crosses the wires of the transit instrument at E, the observer makes the records on both chronographs by tapping his key. When the same star reaches the meridian of W the observer there makes a similar record on both chronographs, and similarly for the other stars. The observers must also make suitable observations for determining the constants of their instruments and the rate of the clock. Let

θ_e = the clock time when a star is on the meridian of E, taken from the chronograph at E,
θ_e' = the same, taken from the chronograph at W,
θ_w = the clock time that the same star is on the meridian of W, taken from the chronograph at E,
θ_w' = the same, taken from the chronograph at W,
e = the personal equation of the observer at E,
w = the personal equation of the observer at W,
$\delta\theta$ = the correction for rate in the interval $\theta_w - \theta_e$.

Then

$$\theta_w + \delta\theta + w - \mu - L = \theta_e + e,$$
$$\theta_w' + \delta\theta + w + \mu - L = \theta_e' + e,$$

Therefore

$$L = \tfrac{1}{2}(\theta_w + \theta_w') - \tfrac{1}{2}(\theta_e + \theta_e') + \delta\theta + w - e,$$

which we may write

$$L = L_1 + w - e. \tag{243}$$

If now the observers exchange places and repeat the observations we shall obtain

$$L = L_2 + e - w, \tag{244}$$

provided their relative personal equations have not changed. Therefore,

$$L = \tfrac{1}{2}(L_1 + L_2). \tag{245}$$

Great care must be taken in arranging the circuits to insure that the constants are the same at both stations, and that the current remains uniform throughout the observations. If these conditions are realized, the results will be free from all errors except the accidental errors of observation.

In case the telegraph line can be used only a few minutes, a set of signals can be sent back and forth in such a way as to be recorded on both chronographs. The time at the two stations having been accurately determined, the results obtained by this method are nearly as accurate as those obtained by star signals.

(c). *By Moon Culminations.* The right ascension of the moon is tabulated in the Nautical Almanac for every hour of Greenwich mean time, whence its value may be computed for any instant of time at a place whose longitude is known. Conversely, if its right ascension is observed at a given place, the Greenwich time corresponding to this right ascension can be taken from the Almanac. The Greenwich time minus the time of observation is the longitude of the observer west of Greenwich.

The right ascension is best observed with a transit instrument in the meridian. An observing list containing two azimuth stars and four or more stars whose declinations are equal to that of the moon as nearly as possible, is arranged so that the moon is near the middle of the list. The transit of the stars and the moon's bright limb are observed in the usual way. From the star transits the constants of the instrument and the chronometer correction at the instant of observing the moon are obtained as before. The distance of the moon's bright limb east of the meridian at the time of observation, θ_m, is given very approximately by [the neglected effect of parallax is small when the instrument is nearly in the meridian]

$$\tau = Aa + Bb + Cc'. \tag{246}$$

The values of A, B and C must be computed for the apparent declination; that is, the geocentric declination minus the parallax, given by (57).

τ is the time required for a star to pass through the angle τ. If Δa is the increase of the moon's right ascension in one mean minute, (given in the Almanac), the mean time required by the

moon to describe the angle τ, is $\tau \dfrac{60}{60 - \varDelta\alpha}$. The sidereal interval is therefore $\tau \dfrac{60.164}{60.164 - \varDelta\alpha}$. Let $M = \dfrac{60.164}{60.164 - \varDelta\alpha}$. The values of log M can be taken from the following table:

$\Delta\alpha$	log M	$\Delta\alpha$	log M	$\Delta\alpha$	log M	$\Delta\alpha$	log M
$1^s.65$	0.0121	$1^s.95$	0.0143	$2^s.25$	0.0166	$2^s.55$	0.0188
1 .70	.0124	2 .00	.0147	2 .30	.0169	2 .60	.0192
1 .75	.0128	2 .05	.0151	2 .35	.0173	2 .65	.0196
1 .80	.0132	2 .10	.0154	2 .40	.0177	2 .70	.0199
1 .85	.0136	2 .15	.0158	2 .45	.0181	2 .75	.0203
1 .90	.0139	2 .20	.0162	2 .50	.0184	2 .80	.0207

The "sidereal time of semidiameter passing meridian" is tabulated in the Nautical Almanac. Let S represent it. The right ascension of the moon's centre when on the meridian is equal to the observer's sidereal time θ, and is given by

$$\alpha = \theta = \theta_m + \Delta\theta + \tau M \pm S, \tag{247}$$

the upper or lower sign being used according as the west or east limb is observed.

Example. The moon's east limb and seven stars were observed with the transit instrument of the Detroit Observatory, 1891 May 23, to determine the longitude.

The star transits gave

$$\begin{aligned} \Delta\theta &= +15^m 34^s.93 \text{ at chronometer time } 16^h 13^m, \\ a &= -0^s.360, \\ b &= +0.674, \\ c' &= +0.100. \end{aligned}$$

The mean of the observed times of transit of the moon's second limb over the five wires was

$$\theta_m = 16^h 12^m 45^s.60.$$

$$\begin{aligned} \text{The moon's geocentric declination} &= -22^\circ\ 3', \\ \text{Parallax} &= 0\ 51\ , \\ \text{The moon's apparent declination} &= -22\ 54\ . \end{aligned}$$

Therefore $A = +0.985$, $B = +0.455$, $C = +1.085$; and $\tau = 0^s.04 = \tau M$. From the Almanac, p. 388, $S = 1^m 9^s.90$. Therefore

$$\alpha = \theta = 16^h 12^m 45^s.60 + 15^m 34^s.93 + 0^s.04 - 1^m 9^s.90 = 16^h 27^m 10^s.67.$$

8

From the Almanac, p. 83, the right ascension at Gr. mean time 18^h was $16^h\ 27^m\ 20^s.32$. The difference, $9^s.65$, corresponds to a difference in time of about 4^m. The average increase of right ascension per minute during this interval was $2^s.3058$. The exact value of the interval before 18^h is $9.65 \div 2.3058 = 4^m.185 = 4^m\ 11^s.10$. The Greenwich mean time corresponding to α was therefore $17^h\ 55^m\ 48^s.90$. The equivalent sidereal time was $22^h\ 2^m\ 0^s.38$, and the longitude of the observer was

$$L = 22^h\ 2^m\ 0^s.38 - 16^h\ 27^m\ 10^s.67 = 5^h\ 34^m\ 49^s.71.$$

Longitudes obtained by this method can be regarded only as approximately correct, for two reasons:

1st. An error in the observed right ascension introduces an error $\frac{60}{\Delta\alpha}$ times as great in the resulting longitude;

2d. The tables of the moon's motion are imperfect. The Greenwich observations for 1890 show that the average correction for the year to Hansen's right ascensions was $+0^s.23$. This would introduce an errror of about $\frac{60}{\Delta\alpha}\ 0^s.23 = 6^s$ in a resulting longitude. The above example should be reduced anew when the corrections to the moon's right ascensions for 1891 are published.

CHAPTER IX.

THE ZENITH TELESCOPE.

111. When a sensitive spirit level at right angles to the rotation axis and a micrometer moving parallel to the axis are added to the transit instrument, it becomes a zenith telescope. The level is called a *zenith level.* The transit instrument and the zenith telescope are usually combined in this way.

The zenith telescope is especially adapted to determining the latitude when great accuracy is required. The method employed is known as *Talcott's method.* It consists in measuring the difference of the zenith distances of two stars, one of which culminates south of the zenith and the other north of the zenith. The difference of their zenith distances should not exceed half the diameter of the field of view, to avoid observing near the edge of the field. The difference of their right ascensions should not exceed 15^m or 20^m, to avoid any change in the con-

stants of the instruments between the two halves of the observation; nor should the difference be less than 2^m or 3^m, to avoid undue haste. The zenith distances should never exceed 35°, to avoid uncertainty in the refractions.

To prepare the observing list an approximate value of the latitude must be known. This can be found from a map, or from a sextant meridian double altitude, (§ 88). Letting the primes refer to the southern star and the seconds to the northern star, we have

$$\delta' = \phi - z', \tag{248}$$
$$\delta'' = \phi + z''. \tag{249}$$

Therefore

$$\delta' + \delta'' = 2\,\phi + (z'' - z'), \tag{250}$$

which is the condition that the two stars of the pair must fulfill. Thus, in latitude 42° 17′, and with an instrument whose field of view is 40′ in diameter, we must have two stars such that $\delta' + \delta''$ is greater than 84° 14′ and less than 84° 54′. A pair is given below which meets these requirements. The "Setting" is the mean of the zenith distances. The assumed latitude is 42° 17′.

Star.	Mag.	Apparent α	δ	z	Setting.
κ *Ursae Majoris*	3.3	$8^h\ 56^m\ 12^s$	+47° 35′	N. 5° 18′	N. 5° 9′
38 *Lyncis*	4.1	9 12 5	37 16	S. 5 1	S. 5 9

Care must be taken in forming the observing list to employ only those stars whose declinations are well determined.

To observe the first star, the circle to which the zenith level is usually attached is made to read the "Setting," the telescope is rotated until the bubble moves to the middle of the tube, and the micrometer wire is moved to the part of the eye piece where it is known the star will pass. Thus, in the pair above, it is known that the first star crosses 9′ [= 5° 18′ — 5° 9′] above the centre. When the first star culminates, or within a few seconds of culmination, as shown by a chronometer and the right ascension, bisect the star by the micrometer wire, read the zenith level and the micrometer. Reverse the instrument, bring the bubble to the centre of the level again, and observe the second star in the same way as the first. It is sometimes pref-

erable not to clamp the instrument during the observations. *Care must be taken not to change the position of the level with respect to the line of sight during the progress of an observation; the angle between the two must be preserved.*

112. Let m_0 be the micrometer reading on any point of the field assumed as the micrometer zero; z_0 the apparent zenith distance corresponding to m_0 when the level bubble is at the centre of the tube; m', m'' the micrometer readings on the two stars, the readings being supposed to increase with the zenith distance; R the value of a revolution of the micrometer screw; b', b'' the level constants for the two stars, plus when the north end is high; r', r'' the refractions for the two stars. Then the true zenith distance of the southern star is given by

$$z' = z_0 + (m' - m_0) R + b' + r';$$

and of the northern star

$$z'' = z_0 + (m'' - m_0) R - b'' + r''.$$

Substituting these in (250) and solving for φ,

$$\phi = \tfrac{1}{2}(\delta' + \delta'') + \tfrac{1}{2}(m' - m'') R + \tfrac{1}{2}(b' + b'') + \tfrac{1}{2}(r' - r''). \qquad (251)$$

If the micrometer readings decrease for increasing zenith distances the sign of the second term is minus.

In case the zero of the level scale is at the centre of the tube,

$$\tfrac{1}{2}(b' + b'') = \tfrac{1}{4}[(n' + n'') - (s' + s'')] d, \qquad (252)$$

in which n', n'', s', s'' are the level readings for the two stars, and d is the value of a division of the level.

In case the zero of the level scale is at one end of the tube,

$$\tfrac{1}{2}(b' + b'') = \tfrac{1}{4}[\pm(n' + s') \mp (n'' + s'')] d, \qquad (253)$$

the upper sign being used when n' is greater than s', the lower when n' is less than s'.

The refraction correction is small and can be computed differentially by the formula

$$\tfrac{1}{2}(r' - r'') = \tfrac{1}{2}\frac{dr}{dz}(z' - z''), \qquad (254)$$

in which $(z' - z'')$ is expressed in minutes of arc, and $\dfrac{dr}{dz}$ is the rate of change of refraction in seconds of arc per minute of change in zenith distance. Differentiating [See p. 23.]

we obtain

$$r = 57''.7 \tan z,$$

$$\frac{dr}{dz} = 57''.7 \sec^2 z \sin 1', \tag{255}$$

the factor, sin 1′, being introduced to make the two members homogeneous. Therefore, from (254),

$$\tfrac{1}{2}(r' - r'') = 28''.85 \sec^2 z \sin 1' (z' - z''). \tag{256}$$

The values of $\frac{1}{2}(r' - r'')$ can be taken from the following table, for the given zenith distance. The sign of this correction is the same as that of the micrometer correction.

Values of $\frac{1}{2}(r' - r'')$.

$z' - z''$	$z = 0°$	$z = 10°$	$z = 20°$	$z = 25°$	$z = 30°$	$z = 35°$
0′	″.00	″.00	″.00	″.00	″.00	″.00
2	.02	.02	.02	.02	.02	.02
4	.03	.03	.04	.04	.04	.05
6	.05	.05	.06	.06	.07	.08
8	.07	.07	.08	.08	.09	.10
10	.08	.09	.10	.10	.11	.13
12	.10	.10	.11	.12	.13	.15
14	.12	.12	.13	.14	.15	.18
16	.13	.14	.15	.16	.18	.21
18	.15	.16	.17	.18	.20	.23
20	.17	.18	.19	.21	.23	.26
22	.18	.19	.21	.23	.25	.28
24	.20	.21	.23	.25	.27	.31

If for any reason the star can not be observed at the instant of culmination, the bisection may be made when the star is at one side, the time of observation being noted. A zenith distance observed in this way is too small for a star south of the zenith and too large for a star north of the zenith. A slight correction, called *the reduction to the meridian,* must be made. Let x represent it; and let t be the hour angle of the star when it was observed, in seconds of time.

In the right triangle formed by the meridian, the star's declination circle, and the micrometer wire projected on the sphere, we have the side $90° - \delta$ and the angle t at the pole, to find the side $90° - (\delta + x)$. We can write

$$\cot(\delta + x) = \cos t \cot \delta.$$

Expanding and solving for tan x,

$$\tan x = \frac{(1 - \cos t) \sin \delta \cos \delta}{\sin^2 \delta + \cos t \cos^2 \delta}.$$

We can put the denominator equal to unity without sensible error, since t is always small. Therefore

$$\tan x = 2 \sin^2 \tfrac{1}{2} t \sin \delta \cos \delta = \tfrac{1}{2} \sin 2\delta \,.\, 2 \sin^2 \tfrac{1}{2} t;$$

or,

$$x = \tfrac{1}{2} \sin 2\delta \, \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}. \tag{257}$$

The correction to the observed latitude will always be $+ \frac{1}{2} x$. If both stars of the pair are observed off the meridian, there will be two such terms to apply.

The values of x are tabulated below with the arguments δ and t.

Values of x.

δ \ t	5^s	10^s	15^s	20^s	25^s	30^s	35^s	40^s	45^s	50^s	55^s	60^s	t / δ
0°	″.00	″.00	″.00	″.00	″.00	″.00	″.00	″.00	″.00	″.00	″.00	″.00	90°
5	.00	.00	.01	.02	.03	.04	.06	.08	.10	.12	.14	.17	85
10	.00	.01	.02	.04	.06	.08	.11	.15	.19	.23	.28	.34	80
15	.00	.01	.03	.05	.09	.12	.17	.22	.28	.34	.41	.49	75
20	.00	.02	.04	.07	.11	.16	.22	.28	.36	.44	.53	.63	70
25	.01	.02	.05	.08	.13	.19	.26	.34	.42	.52	.63	.75	65
30	.01	.02	.05	09	.15	.21	.29	.38	.48	.59	.71	.85	60
35	.01	.03	.06	.10	.16	.23	.31	.41	.52	.64	.77	.92	55
40	.01	.03	.06	.11	.17	.24	.33	.43	.54	.67	.81	.97	50
45	.01	.03	.06	.11	.17	.25	.33	.44	.55	.68	.82	.98	45

ADJUSTMENTS.

113. The adjustments for the transit, § 103, apply equally well, for the most part, to the zenith telescope. Special forms of the instrument, however, will call for special methods, which the intelligent observer will easily devise.

The micrometer wire must be made perpendicular to the meridian. If this adjustment is perfect, an equatorial star will travel on the wire throughout its entire length.

114. *Example.* The following observations were made with the zenith telescope of the Detroit Observatory, 1891 March 16.

Star.	Chronom.	Microm.	Level.	
			n	s
κ *Ursae Majoris*	$8^h\ 40^m\ 36^s$	13.647	8.9	35.7
38 *Lyncis*	8 56 10	37.359	39.6	12.4

Required the latitude.

The chronometer correction was $+15^m 47^s$. The value of one revolution of the micrometer screw was $R = 45''.042$. The value of one division of the level was $d = 2''.74$. The mean places of the stars are given in the *Jahrbuch*, p. 180. Their apparent places are found by the methods of § 54 to be

$$\alpha'' = 8^h\,56^m\,12^s, \qquad \delta'' = +47^\circ\,35'\,21''.33\,,$$
$$\alpha' = 9\;\;12\;\;\;\;5, \qquad \delta' = +37\;\;15\;\;53\;\;.05\,.$$

Therefore

$$\tfrac{1}{2}(\delta' + \delta'') = 42^\circ\,25'\,37''.19\,.$$

The micrometer readings decreased with increasing zenith distances. Therefore

$$-\tfrac{1}{2}(m' - m'')\,R = -11.856\,R = -8'\,54''.02\,.$$

The zero of the level was at one end; therefore, by (253),

$$\tfrac{1}{2}(b' + b'') = \tfrac{1}{4}(52.0 - 44.6)\,d = +5''.07\,.$$

The difference of the zenith distances is $17'$, and the mean zenith distance is $z = 5^\circ\,9'$. From the table for differential refraction,

$$\tfrac{1}{2}(r' - r'') = -0''.14\,.$$

The first star was observed at an hour angle $t = +11^s$; therefore, from the table, the value of $\frac{1}{2}x$ for the northern star is

$$\tfrac{1}{2}x'' = +0''.02\,.$$

The second star was observed at the hour angle $t = -8^s$; therefore the value of $\frac{1}{2}x$ for the southern star is

$$\tfrac{1}{2}x' = +0''.01\,.$$

Combining the terms of (251) and the reductions to the meridian, we obtain

$$\phi = 42^\circ\,16'\,48''.13\,.$$

[The correct value is about $42^\circ\,16'\,47''.3$].

In very accurate determinations of the latitude, a number of pairs should be observed several times in this way, and the results combined by the method of least squares. If we let φ_0, R_0 and d_0 be very nearly the true values of φ, R and d, and let $\Delta\varphi$, ΔR and Δd be slight corrections to φ_0, R_0 and d_0, each observation furnishes an equation of the form

$$\begin{aligned}\phi_0 + \Delta\phi &= \tfrac{1}{2}(\delta' + \delta'') + \tfrac{1}{2}(m' - m'')(R_0 + \Delta R)\\ &+ \tfrac{1}{4}[\pm(n' + s') \mp (n'' + s'')](d_0 + \Delta d) + \tfrac{1}{2}(r' - r'') + \tfrac{1}{2}x' + \tfrac{1}{2}x''.\end{aligned}$$

Let

$$\begin{aligned}k = \phi_0 &- \tfrac{1}{2}(\delta' + \delta'') - \tfrac{1}{2}(m' - m'')\,R_0\\ &- \tfrac{1}{4}[\pm(n' + s') \mp (n'' + s'')]\,d_0 - \tfrac{1}{2}(r' - r'') - \tfrac{1}{2}x' - \tfrac{1}{2}x''. \qquad (258)\end{aligned}$$

Then

$$\Delta\phi - \tfrac{1}{2}(m' - m'')\,\Delta R - \tfrac{1}{4}[\pm(n' + s') \mp (n'' + s'')]\,\Delta d + k = 0 \qquad (259)$$

is an observation equation for determining $\Delta\varphi$, ΔR and Δd. Thus, in the example above, if we assume $\varphi_0 = 42° \, 16' \, 47''.0$, $R_0 = 45''.040$, $d_0 = 2''.70$, we find $k = -1''.08$, and (259) becomes

$$\Delta\phi + 11.856 \, \Delta R - 1''.85 \, \Delta d - 1''.08 = 0. \tag{260}$$

Forming the corresponding equations for the other pairs observed and solving by the method of least squares, the most probable values of $\Delta\varphi$, ΔR and Δd, and therefore of φ, R and d, are obtained.

The weights to be assigned to the different observations depend so largely on the star catalogues used in determining the declinations, that no satisfactory directions can be given here. A knowledge of the subject of star catalogues is required. A table of relative weights given in the introduction to *Newcomb's Standard Stars* will serve as a partial guide.

CHAPTER X.

ASTRONOMICAL AZIMUTH.

115. In many problems of the higher surveying the azimuth of a point on the earth's surface is required to be very accurately known. It is determined by measuring the *difference* of the azimuths of the point and a star by means of a theodolite, a surveyor's transit, or any other instrument designed for measuring horizontal angles. The azimuth of the star is computed from the known right ascension, declination, latitude and time; whence the azimuth of the point can be obtained.

Only the four circumpolar stars, whose places are given in the Nautical Almanac, pp. 302–313, should be used in accurate determinations.

The point whose azimuth is to be determined is marked conveniently by a lamp arranged to shine through a small hole in a box, placed directly over the point. It should be at least one mile from the observer. If no provision has been made for illuminating the wires of the telescope at night, they can be rendered visible by tying a piece of thin unglazed white paper over the object end of the telescope, first cutting a hole in it nearly as large as the object glass, and throwing the light of a bull's-eye lantern on the paper.

The instrument is first carefully adjusted, then set up over

the point of observation (marked in some way) and leveled. The horizontal graduated circle is *fixed* in position by clamping. The level screws and other adjusting screws must not be touched *during* a series of observations. If the rotation axis of the telescope is not truly horizontal, an error is introduced in the measured difference of azimuth of the mark and star, which must be allowed for. In the finer instruments the inclination of the axis is measured by means of a striding level. The effect of an error of collimation is practically eliminated by reversing the instrument and observing an equal number of times in both positions; and we shall consider that this is always done. If more than one series of observations is made, the horizontal circle should be shifted so that a different part of it may be used, thereby eliminating largely any errors of graduation of the circle.

116. *Correction for Level.* When the rotation axis of the telescope is inclined to the horizon, the line of sight does not describe a vertical circle, and the horizontal circle reading requires a small correction. Let b be the elevation of the west end of the axis above the horizon, and let the west end of the produced axis cut the celestial sphere in W; let y be the corresponding correction to the circle reading; and let Z be the zenith and S the star. Then, in the triangle WZS

$$WZ = 90° - b, \qquad ZS = z, \qquad WS = 90°, \qquad WZS = 90° + y,$$

and, therefore,

$$\sin b \cos z - \cos b \sin z \sin y = 0.$$

But b and y are very small, and we may write

$$y = b \cot z; \tag{261}$$

and for circumpolar stars we may write

$$y = b \tan \phi. \tag{262}$$

The value of b is found by (144) or (145), or by § 98. If the illuminated mark is not in the horizon, the circle readings on the mark must be corrected by (261), using its zenith distance z.

117. *Correction for diurnal aberration.* Owing to the diurnal aberration the star is observed too far east. In the most refined observations this must be allowed for. The correction to the circle reading is given by (102); which, for circumpolar stars, is approximately

$$dA = +0''.31 \cos A. \tag{263}$$

If the circles cannot be read to less than 1″, this correction is negligible.

118. *Correction for error of runs.* If reading microscopes are used [§ 57], the circle readings must be further corrected for error of runs.

AZIMUTH BY A CIRCUMPOLAR STAR NEAR ELONGATION.

119. A star is at *western* or *eastern elongation* when its azimuth is the least or greatest possible. It is then moving in a vertical circle, and is in the most favorable position for azimuth observations. Only one observation can be made *at* the instant of elongation, however, and it is customary to make several observations just before and after elongation, and allow for the change in azimuth during the intervals.

At the instant of elongation the triangle formed by the pole, star and zenith, which we shall denote by PSZ, is right-angled at the star. If we let θ_0 be the sidereal time, and A_0, t_0 and z_0 the azimuth, hour angle and zenith distance of the star at elongation, α and δ its right ascension and declination, and φ the observer's latitude, we shall have, for western elongation,

$$PZ = 90^\circ - \phi, \qquad PS = 90^\circ - \delta, \qquad ZS = z_0,$$
$$PZS = 180^\circ - A_0, \qquad ZPS = t_0, \qquad PSZ = 90^\circ;$$

and for eastern elongation,

$$PZS = A_0 - 180^\circ, \qquad ZPS = 360^\circ - t_0.$$

We can write

$$\cos t_0 = \frac{\tan \phi}{\tan \delta}, \qquad \cos z_0 = \frac{\sin \phi}{\sin \delta}, \qquad \sin A_0 = \pm \frac{\cos \delta}{\cos \phi}, \tag{264}$$

$$\theta_0 = \alpha + t_0. \tag{265}$$

t_0 is in the first quadrant for western elongation, and in the fourth for eastern; z_0 is always in the first quadrant; and A_0 is in the second quadrant for western elongation, and in the third for eastern. We can also write

$$\pm \sin A_0 = \frac{\cos \delta}{\cos \phi} = \frac{\sin \delta \cos t_0}{\sin \phi}, \tag{266}$$

$$\pm \cos A_0 = -\sin \delta \sin t_0, \tag{267}$$

in which the upper signs are for western elongation, the lower for eastern.

If the star is observed at any other hour angle t, its azimuth A is given by (16) and (17). Multiplying (16) by (266), (17) by (267), and subtracting one product from the other, we obtain

$$\sin z \sin (A_0 - A) = \mp \sin \delta \cos \delta \; 2 \sin^2 \tfrac{1}{2} (t_0 - t). \tag{268}$$

If the observations are made near elongation, t will not differ much from t_0, $A_0 - A$ will be small, and for the circumpolar stars z will not differ much from z_0; and we can write, without sensible error,

$$A_0 - A = \mp \frac{\sin \delta \cos \delta}{\sin z_0} \cdot \frac{2 \sin^2 \frac{1}{2} (t_0 - t)}{\sin 1''}, \tag{269}$$

in which the lower sign is for eastern elongation, as before. $A_0 - A$ is the correction to be applied to the circle reading for an observation made at hour angle t to reduce to the corresponding circle reading for an observation made at hour angle t_0.

For convenience, let

$$m = \frac{\sin 2^2 \frac{1}{2} (t_0 - t)}{\sin 1''},$$

and (269) becomes

$$A_0 - A = \mp m \frac{\sin \delta \cos \delta}{\sin z_0}. \tag{270}$$

The values of m can be taken from Table III, for the different values of $t_0 - t$. If we let m_0 be the mean of the several values of m, the corrections can be applied collectively to the mean of the circle readings on the star, and the equation (270) becomes

$$A_0 - A = \mp m_0 \frac{\sin \delta \cos \delta}{\sin z_0}, \tag{271}$$

in which $A_0 - A$ is the correction to the mean of the circle readings. Further, if the level readings have been taken symmetrically, which can always be done, the mean value of y, equation (262), can be applied to the mean of the circle readings.

120. The values of t_0 and θ_0 having been computed for a certain star, the instrument is set firmly in position and leveled, and a program similar to this is followed:

Make two readings on the mark.
Read the level.
Make four readings on the star.
Read the level.
Make two readings on the mark.
Reverse.
Make two readings on the mark.
Read the level.
Make four readings on the star.
Read the level.
Make two readings on the mark.

The times of observation are noted on a time-piece, preferably a sidereal chronometer; and its correction must be known within one or two seconds if the most refined form of instrument is employed, or to the nearest minute if an ordinary surveyor's transit is used. This correction can be obtained by any of the methods described in the preceding chapters, or by a comparison with the time signals at the nearest telegraph station. The chronometer time of elongation is now known. Subtracting from it the several times of observation, the values of $t_0 - t$ are found, and the values of m corresponding to them can be taken from Table III. Forming the mean m_0 and computing z_0 from (264), the value of $A_0 - A$ can be found and applied to the mean of the circle readings. The mean of the corrections for level errors and the correction for diurnal aberration are now applied. The corrected mean circle reading, which we shall call s, corresponds to the azimuth A_0 of the star at elongation which is computed by (264). If k is the mean of the circle readings on the mark, and M the azimuth of the mark, then

$$M = k - (s - A_0). \tag{272}$$

121. *Example.* Detroit Observatory, Wednesday, 1891 May 6. Find the azimuth of a given point (nearly in the horizon) from observations on δ *Ursae Minoris* near its eastern elongation. Observer's latitude, 42° 16′ 48″.

The apparent place of the star was

$$\alpha = 18^h\ 7^m\ 44^s, \qquad \delta = +86^\circ\ 36'\ 25''.0.$$

The solution of (264) and (265) is as below.

$\tan\phi$ 9.958704	$\sin\phi$ 9.827856	$\cos\delta$ 8.772214
$\tan\delta$ 1.227024	$\sin\delta$ 9.999236	$\cos\phi$ 9.869153
t_0 273° 5′ 25″	z_0 47° 37′ 42″	A_0 184° 35′ 17″.8
t_0 $18^h\ 12^m\ 22^s$		
α 18 7 44		
θ_0 12 20 6		

The chronometer correction was $+18^m\ 52^s$, and, therefore, the chronometer time of elongation was $12^h\ 1^m\ 14^s$. A good surveyor's transit, reading to 10″, and provided with plate levels and striding level, was placed over the point of observation and carefully leveled and the plate clamped, about half an hour before elongation. The following observations were made:

No.	Object.	Telescope.	Chronom.			Vernier A.			Vernier B.		
(1)	Mark.	Reversed.				96°	16′	40″	276°	16′	30″
(2)	"	"				96	16	35	276	16	25
(3)	Level.										
(4)	Star.	"	11^h	44^m	52^s	243	39	20	63	39	20
(5)	"	"		48	40	243	39	50	63	39	50
(6)	"	"		51	6	243	39	50	63	39	50
(7)	"	"		53	11	243	40	0	63	40	5
(8)	Level.										
(9)	Mark.	"				96	16	45	276	16	40
(10)	"	"				96	16	50	276	16	40
(11)	"	Direct.				276	17	0	96	16	45
(12)	"	"				276	16	55	96	16	45
(13)	Level.										
(14)	Star.	"	12	5	50	63	40	10	243	40	10
(15)	"	"		7	54	63	40	0	243	40	0
(16)	"	"		9	44	63	39	50	243	39	50
(17)	"	"		11	27	63	39	45	243	39	55
(18)	Level.										
(19)	Mark.	"				276	16	45	96	16	30
(20)	"	"				276	16	45	96	16	35

The level readings given by the striding level were

(3) W	(3) E	(8) W	(8) E	(13) W	(13) E	(18) W	(18) E
4.4	4.1	4.2	4.0	4.0	4.4	4.0	4.2
4.3	4.0	4.3	3.8	4.0	4.2	4.1	4.2
4.4	4.0	4.2	3.9	4.0	4.2	4.0	4.2
4.3	3.8	4.4	3.8	4.1	4.4	4.2	4.2

The value of one division of the level was 10″.7, and therefore from (144), the inequality of the pivots being negligible,

$$b = \overset{(3)}{+2''.0} \qquad \overset{(8)}{+2''.1} \qquad \overset{(13)}{1''.4} \qquad \overset{(18)}{0''.6},$$

and by (262),

$$y = \overset{(3)}{+1''.8} \qquad \overset{(8)}{+1''.9} \qquad \overset{(13)}{-1''.3} \qquad \overset{(18)}{-0''.6}.$$

The solution of (271) for the eight readings is given below. The column "Circle Readings" is formed by taking the means of Vernier A and Vernier B.

No.	Circle Readings.			$t_0 - t$		m
(4)	243°	39′	20″	$+16^m$	22^s	525″.7
(5)	243	39	50	+12	34	310 .0
(6)	243	39	50	+10	8	201 .6
(7)	243	40	2	+ 8	3	127 .2
(14)	63	40	10	− 4	36	41 .5
(15)	63	40	0	− 6	40	87 .3
(16)	63	39	50	− 8	30	141 .8
(17)	63	39	50	−10	13	204 .9
	63	39	51.5		$m_0 =$	205 .0

$\log m_0$	2.31175
$\sin \delta$	9.99924
$\cos \delta$	8.77221
$\operatorname{cosec} z_0$	0.13148
$\log (A_0 - A)$	1.21468
$A_0 - A$	+16″.4

The mean of the four values of y is $+0''.4$. The values of dA, from (263), is $-0''.3$. The corrected circle reading on the star at elongation is therefore

$$s = 63^\circ\ 39'\ 51''.5 + 16''.4 + 0''.4 - 0''.3 = 63^\circ\ 40'\ 8''.0.$$

The mean of all the readings on the mark is

$$k = 276^\circ\ 16'\ 41''.6;$$

and, therefore, by (272),

$$M = 37^\circ\ 11'\ 51''.4.$$

Since the verniers on this instrument read to only 10″, the diurnal aberration could be neglected, and the other corrections computed to the nearest second only. But all the corrections have been applied here to illustrate the method.

AZIMUTH BY POLARIS OBSERVED AT ANY HOUR ANGLE.

122. When the azimuth is not wanted with the greatest possible accuracy, good results can be obtained by observing *Polaris* in any position, in case the time is known within 1^s for the finest instrument, and within 5^s or 10^s for a good surveyor's transit. As before, the observations should be taken on the mark and star in both positions of the telescope. If the observations are made in quick succession the mean of two or three observations made *before* reversing may be treated as a single observation, and similarly for those made *after* reversing. The correction for level is given as before by (261), and the diurnal aberration by (263).

The time of observation having been noted, the hour angle of *Polaris* is given by (40), and the azimuth and zenith distance by (20), (21) and (22). A will be in the second quadrant if t is between 0^h and 12^h, and in the third quadrant if t is between 12^h and 24^h. z will always be in the first quadrant. The azimuth of the mark will be given, as in (272), by

$$M = k - (s - A). \tag{273}$$

A good program to follow is that given in the observations below.

Example. Detroit Observatory, Wednesday, 1891 May 6 Find the azimuth of a given point nearly in the horizon, from the following observations of *Polaris*, made with the instrument described in § 121.

No.	Object.	Telescope.	Chronom.	Vernier A.	Vernier B.
(1)	Mark.	Direct.		276° 16′ 40″	96° 16′ 35″
(2)	"	"		276 16 40	96 16 30
(3)	Level.				
(4)	Star.	"	13^h 22^m 59^s	59 15 50	239 15 40
(5)	"	"	13 26 30	59 17 0	239 16 50
(6)	"	"	13 27 57	59 17 30	239 17 20
(7)	"	Reversed.	13 32 47	239 19 40	59 19 45
(8)	"	"	13 34 56	239 20 35	59 20 30
(9)	"	"	13 36 40	239 21 20	59 21 15
(10)	Level.				
(11)	Mark.	"		96 16 40	276 16 30
(12)	"	"		96 16 40	276 16 35

The striding level gave

(3)		(10)	
W	E	W	E
3.6	5.3	4.9	4.1
4.6	4.3	4.5	4.5

whence, by (144),

$$\begin{array}{lcr} & (3) & (10) \\ b = & -3''.3 & +2''.1. \end{array}$$

The position of *Polaris* was, Nautical Almanac, p. 306,

$$a = 1^h\ 17^m\ 53^s, \qquad \delta = 88^\circ\ 43'\ 29'',$$

and the chronometer correction was $+18^m\ 52^s$.

The means of the chronometer times of observation before and after reversal give

	Before.	After.
Chronometer	13^h 25^m 49^s	13^h 34^m 48^s
$\Delta\theta$	+ 18 52	+ 18 52
θ	13 44 41	13 53 40
a	1 17 53	1 17 53
$\theta - a = t$	12 26 48	12 35 47
t	186° 42′ 0″	188° 56′ 45″

Solving (20), (21) and (22) for these two values of t, using $\varphi = 42^\circ\ 16'\ 48''$, we find

	Before.	After.
N	91° 16′ 0″	91° 15′ 35″
A	180 11 50	180 15 46
z	48 59	48 59

The corrections for level are now found from (261) to be $y = -3''$ and $y = +2''$.

Taking the means of the circle readings in the two positions we find

	Before.	After.
Mean on star	59° 16′ 42″	239° 20′ 31″
y	− 3	+ 2
s	59 16 39	239 20 33
k	276 16 36	96 16 36
A	180 11 50	180 15 46
M	37 11 44	37 11 49

The mean of the two values is

$$M = 37° \ 11' \ 46''.$$

CHAPTER XI.

THE SURVEYOR'S TRANSIT.

123. The surveyor's transit is adapted to the determination of the time, latitude and azimuth by many of the preceding methods. They can easily be determined to an accuracy within the least readings of the circles, if the instrument is of reliable make and is provided with trustworthy spirit levels. We shall assume that the observer uses a *mean* timepiece, which we shall call a watch, and that he has a thorough knowledge of the subject of Time, Chap. II, without which the Nautical Almanac cannot be used intelligently. We shall assume, also, that the vertical circle of his instrument is complete, and that the degrees are numbered consecutively from 0 to 360. In case they are not, the observer can readily reduce his readings to that system. The instrument is supposed to be carefully adjusted. A method of illuminating the wires at night is given in § 115.

DETERMINATION OF TIME.

124. *By equal altitudes of a star.* Set the instrument up firmly, level it, and direct the telescope to a known bright star in the east or southeast. Pointing the telescope slightly *above* the star, clamp the vertical circle and note the time T' when the star crosses the horizontal wire. The vertical circle must not be unclamped. A short time before the star reaches the same altitude west of the meridian, level the instrument, move it in azimuth until the telescope is directed to a point just below the star, wait for the star to enter the field, and note the

time T'' when it crosses the horizontal wire. The sidereal time θ when the star was on the observer's meridian equals its right ascension α, and this corresponds to the mean of the two watch times. Converting the sidereal time $\theta = \alpha$ into the corresponding mean time T, the watch correction ΔT is given by

$$\Delta T = T - \tfrac{1}{2}(T' + T''). \tag{274}$$

Example. Thursday, 1891 March 5. In longtitude $5^h\ 34^m\ 55^s$ *Regulus* was observed at the same altitudes east and west of the meridian at the watch times

$$T' = 8^h\ 7^m\ 34^s, \qquad T'' = 14^h\ 10^m\ 20^s.$$

Required the watch correction.

From the Nautical Almanac, p. 332, $\alpha = \theta = 10^h\ 2^m\ 35^s$. Converting this into mean time, § 12, we find

$$T = 11^h\ 8^m\ 3^s\,;$$

and, therefore, by (274), the watch correction is

$$\Delta T = -54^s\,;$$

or the watch was 54^s too fast.

125. *By a single altitude of a star.* Level the transit. Direct the telescope very slightly above a known star nearly in the east or below a known star nearly in the west, and clamp the telescope. Note the watch time when the star crosses the horizontal wire and read the vertical circle. Unclamp the telescope and repeat the observation once or twice as quickly as possible. Double reverse the instrument and make the *same number* of observations as before. Form the means of the circle readings made before reversal and those made after. Subtract one from the other in that order which makes their difference less than 180°. One half this difference is the apparent zenith distance of the star at T', the mean of the several watch times of observation. Adding to this the refraction given by

$$r = 58'' \tan z, \tag{275}$$

the result is the true zenith distance z. Substituting the values of z, φ and δ in (38), the hour angle t is found. The sidereal time θ is given by, § 18,

$$\theta = a + t. \tag{276}$$

Converting θ into the mean time T, the watch correction is given by

$$\Delta T = T - T'. \tag{277}$$

9

Example. 1891 April 25. In latitude $+42° 16' 47''$ and longtitude $5^h 34^m 55^s$, the following altitudes of *Arcturus* were observed east of the meridian. Find the watch correction.

Telescope.	Watch.	Circle reading.
Direct	$7^h 52^m 23^s$	$34° 43' 0''$
"	53 33	34 55 30
"	54 20	35 4 0
Reversed	55 37	144 42 30
"	56 30	144 33 0
"	57 18	144 24 0

The means of the circle readings are $34° 54' 10''$ and $144° 33' 10''$; and one half their difference is the

Apparent zenith distance,	$54° 49' 30''$	log 58	1.7634
Refraction, r,	1 22	$\tan z$	0.1520
True zenith distance, z,	54 50 52	$\log r$	1.9154
		r	$82''$

From the Nautical Almanac, p. 340,

$$\alpha = 14^h 10^m 43^s, \qquad \delta = +19° 44' 52''.$$

The solution of (38) is

z	$54° 50' 52''$	$\log \sin \frac{1}{2} [z + (\phi - \delta)]$	9.79595
ϕ	$+ 42$ 16 47	$\log \sin \frac{1}{2} [z - (\phi - \delta)]$	9.44449
δ	$+ 19$ 44 52	$\log \sec \frac{1}{2} [z + (\phi + \delta)]$	0.28114
$\phi - \delta$	22 31 55	$\log \sec \frac{1}{2} [z - (\phi + \delta)]$	0.00085
$\phi + \delta$	62 1 39	$\log \tan^2 \frac{1}{2} t$	9.52243
$z + (\phi - \delta)$	77 22 47	$\log \tan \frac{1}{2} t$	9.76121_n
$z - (\phi - \delta)$	32 18 57	$\frac{1}{2} t$	$150° 0' 48''$
$z + (\phi + \delta)$	116 52 31	t	300 1 36
$z - (\phi + \delta)$	$-$ 7 10 47	t	$20^h 0^m 6^s$

Solving (276), $\theta = 10^h 10^m 49^s$. The equivalent mean time is $T = 7^h 55^m 45^s$; and the mean of the six times of observation is $T' = 7^h 55^m 2^s$. Therefore,

$$\Delta T = +43^s.$$

126. *By a single altitude of the sun.** Observe the transits of the sun's upper and lower limbs over the horizontal wire by the method used for a star, § 125. Double reverse and repeat the observations. Take half the difference of the means of the circle readings and add the refraction given by (275), as before. Further, subtract the parallax given by

$$p = 9'' \sin z, \tag{278}$$

* The observer must cover the eye piece with a small piece of very dense red glass before looking through at the sun. The observations can be made, also, by holding a piece of paper a short distance back from the eye piece, and focusing the eye piece so that the images of the sun and wire are seen on the paper.

and the result is the sun's true zenith distance z at the mean of the times, T'. The correct mean time is probably known within 5^m or 10^m. Increase it by the longitude, and the result is an approximate value of the Greenwich mean time. Take from the Nautical Almanac, p. II, of the month, the value of the sun's declination δ at that time. The Almanac contains the apparent declination for Greenwich mean noon, and the "difference for one hour," whence the declination at any instant can be found. Solve (38) for these values of z, δ and φ. The resulting hour angle of t is the observer's *apparent solar time* Convert this into the equivalent mean time T, by § 9. The watch correction is given by (277), as before.

Example. Thursday morning, May 7, 1891. In latitude $+42° 16' 50''$ and longitude $5^h 35^m$, the following observations of the sun were made with Buff & Berger transit No. 1554. Required the watch correction.

Telescope.	Limb.	Watch.	Circle reading.
Direct	Upper	$20^h 38^m 59^s$	$41° 48' 30''$
"	Lower	41 59	41 48 30
Reversed	Upper	46 49	137 33 0
"	Lower	49 50	137 33 0

One half the difference of the circle readings is $47° 52' 15'$. The refraction, by (275), is $64''$, and the parallax, by (278), is $7''$. Therefore the true zenith distance z of the sun's center is $47° 53' 12''$. The mean of the four watch times is $20^h 44^m 24^s$ We have

T'	1891 May	$6^d 20^h 44^m 24^s$
Longitude		5 35
Gr. mean time		7 2 19
" " "		7 $2^h.32$

From the Nautical Almanac, p. 75, the sun's declination at Gr. mean noon, May 7, was $+16° 48' 53''$, and the difference for one hour, $+41''$. The change for $2^h.32$ was therefore $96''$, and the required value of the declination was $\delta = +16° 50' 29''$.

Substituting the values of z, φ and δ in (38), and solving as was done in § 125, we obtain the hour angle $t = 312° 12' 10'' = 20^h 48^m 49^s$. The observer's true time is therefore May $6^d 20^h 48^m 49^s$. Converting this into the mean time T, by § 9, we find $T = 1891$ May $6^d 20^h 45^m 15^s$. The watch correction is

$$\Delta T' = 20^h 45^m 15^s - 20^h 44^m 24^s = +51^s.$$

GEOGRAPHICAL LATITUDE.

127. *By a meridian altitude of a star.* A star is on the observer's meridian when the sidereal time θ is equal to its right ascension α. Convert this into the corresponding mean time, subtract the watch correction obtained by any of the above methods from it, and the result is the watch time of the star's meridian passage. A few seconds before this watch time direct the telescope to the star, bring the star's image on the horizontal wire, and read the circle. Double reverse quickly and make another observation. As before, take one-half the difference of the circle readings, add the refraction given by (275), and the sum is the star's true zenith distance z. Take the value of δ from the Nautical Almanac. For a star observed south of the zenith

$$\phi = \delta + z; \tag{279}$$

and for a star observed between the zenith and pole,

$$\phi = \delta - z. \tag{280}$$

For a star below the pole the sidereal time of meridian passage is $12^h + \alpha$. Obtaining the value of z as before, the latitude is given by

$$\phi = 180^\circ - \delta - z. \tag{281}$$

Example. Ann Arbor, 1891 April 24. α *Hydrae* was observed on the meridian with a transit, as below.

Telescope.	Circle reading.
Reversed	140° 26′ 30″
Direct	39 33 0

One half the difference of the circle readings is 50° 26′ 45″. The refraction is 70″. Therefore, $z = 50^\circ\ 27'\ 55''$. From the Nautical Almanac, p. 331, $\delta = -8^\circ\ 11'\ 18''$. Therefore, from (279),

$$\phi = 42^\circ\ 16'\ 37''.$$

To find the watch time when the star is on the meridian, we have, from the Nautical Almanac, $\alpha = \theta = 9^h\ 22^m\ 14^s$. The corresponding mean time is $7^h\ 11^m\ 13^s$. The watch correction is $+ 43^s$, whence the required watch time is $7^h\ 10^m\ 30^s$.

128. *By a meridian altitude of the sun.* The sun is on the meridian at the apparent time $0^h\ 0^m\ 0^s$. Apply the equation of time to this, by § 9, and subtract the known watch correction. The result is the watch time of the sun's meridian passage. One or two minutes before this watch time, direct the horizontal

wire of the telescope to the upper limb of the sun and read the vertical circle. Observe the lower limb in the same way. Double reverse and observe both limbs again. Take half the difference of the means of the readings in the two positions. Add the refraction given by (275) and subtract the parallax given by (278). The result is the value of z. Take from the Nautical Almanac the value of δ for the time of meridian passage. The latitude is now given by (279), as in the case of a star.

Example. Wednesday, March 25, 1891. In longitude $5^h 35^m$ the following meridian altitude observations of the sun were made with a transit. Required the latitude.

Telescope.	Limb.	Circle reading.
Direct	Upper	49° 54′ 30″
"	Lower	49 22 30
Reversed	"	130 5 30
"	Upper	130 38 30

One half the difference of the means of the circle readings is 40° 21′ 45″. The refraction is 49″. The parallax is 6″. Therefore, $z = 40° 22' 28''$.

The Greenwich apparent time of observation was March $25^d\ 5^h\ 35^m$. The value of δ at that instant was $+1° 54' 32''$, Nautical Almanac, p. 38. Therefore, by (279),

$$\phi = 42° 17' 0''.$$

To find the watch time of meridian passage, we have,

Apparent time	$0^h\ 0^m\ 0^s$
Equation of time	$+6 \quad 3$
Mean time	$0 \quad 6 \quad 3$
Watch correction	$-0 \quad 15$
Watch time	$0 \quad 6 \quad 18$

AZIMUTH.

129. The two methods of determining azimuth which are described in the preceding chapter are adapted to the surveyor's transit, and need no further explanation. But with this instrument the diurnal aberration can be neglected.

If the transit is provided with plate levels only, they should be kept in perfect adjustment. If the bubble of that level which is parallel to the rotation axis of the telescope remains constantly in the centre, no correction for level is required. But if the bubble is n divisions of the level from the centre

when an observation on a star is being made, and d is the value of one division of the level, the circle reading must be corrected by

$$y = \pm n\, d \cot z; \tag{282}$$

+ if the bubble is too far west, — if too far east.

CHAPTER XII.

THE EQUATORIAL.

130. This instrument consists essentially of the following parts: A supporting *pier;* a *polar* or *hour axis* parallel to the earth's axis, supported at two points by the pier in such a way that it can rotate; a *declination axis* attached to the upper end of the hour axis, and at right angles to it, in such a way that it can rotate; a *telescope* firmly attached to one end of the declination axis, and at right angles to it; a graduated *declination circle* attached to the other end of the declination axis; a graduated *hour circle* attached to the polar axis, and at right angles to it; a finding telescope or *finder* to assist in pointing the principal telescope, and attached to it; a *driving clock* and train of wheels for rotating the instrument about its polar axis at a uniform rate. A sidereal chronometer is also an indispensable companion of the equatorial. The various moving parts are so counterpoised that the telescope will be in equilibrium in all positions.

The equatorial serves two purposes:

1st. As an instrument of direct observation and discovery, by assisting the vision.

2nd. As an instrument for determining very accurately the relative positions of two objects comparatively near each other, by means of a micrometer eye piece. If the position of one of the objects is known, the position of the other is known as soon as their *relative* positions are determined.

131. By the above system of mounting it is evident that the telescope can be directed to any part of the sky; and that it will follow a star in its diurnal motion by revolving the instrument about the polar axis alone, for in that case the line of sight maintains a constant angle with the celestial equator, and therefore describes a circle which is identical with the star's diurnal circle. Since the star's motion is uniform, the

telescope is made to follow it by means of the sidereal driving clock. In some observations the driving clock is not used; in others it is indispensable.

When the telescope is revolved upon the declination axis it describes an hour or declination circle.* The position of this hour circle is indicated by the reading of the graduated hour circle. The position of the telescope in this hour circle is indicated by the reading of the graduated declination circle. When the telescope is directed exactly to the south point of the equator, the hour circle reading should be $0^h\ 0^m\ 0^s$, and the declination circle reading should be $0°\ 0'\ 0''$. Then if the other parts of the instrument are in adjustment, and the telescope be directed to a star, the hour angle and declination of the star will be indicated, (neglecting the refraction), by the hour circle and declination circle readings.†

ADJUSTMENTS.

132. It is not essential that the errors of adjustment of an equatorial be eliminated or that their values be accurately known; but it is a practical convenience to have the errors small.

It is expected that the maker of the instrument will adjust the various parts of it as perfectly as possible *with reference to each other*. It remains for the observer to place the instrument in correct position.

The polar axis should be in the plane of the meridian; the elevation of the polar axis should equal the latitude of the place; the hour circle should read zero when the telescope is in the meridian; the declination circle should read zero when the telescope is in the equator; and the lines of sight of the finder and telescope should be parallel.

The instrument should first be placed as nearly as possible in position by estimation. Then direct the telescope to any known star near the southern horizon whose right ascension is α. The star will be on the meridian at the sidereal time $\theta = \alpha$. Move the whole instrument in azimuth so that the star is in the centre

* The hour and declination circles of the celestial sphere must not be confused with the hour and declination circles of the instrument.

† The hour circle will read *time*. It may be graduated from 0^h to 24^h toward the west, or from 0^h to 12^h in both directions from 0^h. The declination circle will read *arc*. It may be graduated from 0° to 360°, or from 0° to 180° in both directions. Neither system should cause the observer any inconvenience. We shall suppose it reads from 0° to 360

of the telescope when the chronometer time plus the chronometer correction is equal to $\theta - \alpha$. The order of making the final adjustments and the simplest methods are as below.

133. *To determine the angle of elevation of the polar axis and the index correction of the hour circle.** Across the object end of the telescope firmly tie a piece of wood which projects several inches from the telescope tube on the side opposite the pier. Pass a fine thread through a very small hole in the projecting end and fasten it. Direct the telescope to the zenith. Near the eye end and on the same side as the projecting arm above fasten a block of wood. To this screw a metal plate so that it will be perpendicular to the axis of the tube, and in which is a very small circular hole as nearly as possible (by estimation) under the hole above. Pass the thread through it, tie a plumb bob to the end of the thread near the floor, and let it swing in a vessel of water. Move the telescope by the slow motion screws until the plumb-line passes through the centre of the lower hole. Read both verniers of the hour and declination circles. Unclamp, hold the plumb bob in the hand to avoid displacing the metal plate, reverse the telescope to the other side of the pier, and set it so that the plumb-line again passes centrally through the hole. Read both circles as before.

Let h equal the angle of elevation of the polar axis. The difference of the readings of the declination circle in the two positions is $180° - 2h$. The elevation should equal the known latitude ϕ. The error is $h - \phi$. Change the last circle reading by this amount by moving the telescope in declination in the proper direction. Adjust the angle of elevation by the proper screws until the plumb-line again passes through the centre of the hole.

The mean of the hour circle readings in the two positions is the reading of the circle when the telescope is in the meridian. This should be $0^h\ 0^m\ 0^s$. The error is the mean of the readings minus $0^h\ 0^m\ 0^s$ (or minus $24^h\ 0^m\ 0^s$). To correct for it, set the telescope to read this mean reading; then move the vernier screws until the reading is $0^h\ 0^m\ 0^s$.

The index correction of the hour circle is equal to the index error with its sign changed. If the error is not removed by adjusting the verniers the index correction must be applied to

*This very simple and satisfactory method was proposed by Professor Schaeberle in the *Astronomische Nachrichten*, No. 2374.

every reading made with the hour circle, in order to obtain the true reading.

If the errors were large these adjustments should be repeated once or twice.

Example. 1891 Feb. 20. The following plumb-line observations were made on the 6-inch equatorial of the Detroit Observatory. Determine the errors. The last column gives the position of the telescope with reference to the pier.

Hour Circle.		Declination Circle.		Telescope.
Vernier A.	Vernier B.	Vernier A.	Vernier B.	
$24^h\ 2^m\ 53^s$	$12^h\ 2^m\ 58^s$	135° 45′ 30″	315° 45′ 00″	E
11 56 56	23 57 6	40 16 45	220 16 30	W

The means of the declination circle readings were 135° 45′ 15″ and 40° 16′ 37″, and therefore h equaled 42° 15′ 41″. The value of φ is 42° 16′ 47″. The axis was therefore 1′ 6″ too low. The telescope was moved in declination until the verniers read 40° 17′ 45″ and 220° 17′ 30″, and the axis adjusted until the thread was again central in the hole.

The hour circle readings were $24^h\ 2^m\ 55^s.5$ and $23^h\ 57^m\ 1^s$, and their mean was $23^h\ 59^m\ 58^s.2$. The error was therefore $-1^s.8$. The verniers were moved to the west 2^s.

A repetition of the observations gave $h = 42°\ 16'\ 49''$, and the mean of the hour circle readings, $25^h\ 0^m\ 0^s.5$. Further adjustment was not required. The index error of the hour circle was $+0^s.5$, and the index correction to be applied to future readings was $-0^s.5$.

134. *To adjust the finder.* Using the lowest power eye piece, direct the telescope to a bright star. Replace the low power eye piece by a high power. Keeping the star in the centre of the field of view, turn the adjusting screws of the finder so that the star is on the intersection of the cross wires in the finder. The two telescopes are then sufficiently near parallelism.

135. *To determine the azimuth correction of the vertical plane containing the polar axis.* This is best determined by observations on one of the four Nautical Almanac circumpolar stars near its culmination.

Using the micrometer eye piece (§§ 59 and 142), direct the telescope to the star a few minutes before its culmination, note the chronometer time θ_1 when the star is on the point of intersection of the wires (or any well defined point in the eye piece), and read the hour circle. Reverse the telescope to the other side of the pier, note the time θ_2 when the star is at the same point of the eye piece, and read the hour circle. Let t_1 and t_2 be the hour circle readings in the two positions, corrected for index error, if any; let a be the required azimuth correction; and let $\Delta\theta$ be the known chronometer correction [see §§ 66 and 138].

Neglecting the quantities which are eliminated by the reversal, we have for the sidereal times when the star is in the vertical plane of the polar axis,

$$\theta_1 + \Delta\theta - t_1,$$
$$\theta_2 + \Delta\theta - t_2.$$

Therefore, as with the transit instrument in § 101,

$$aA = \alpha - (\theta_1 + \Delta\theta - t_1),$$
$$aA = \alpha - (\theta_2 + \Delta\theta - t_2),$$

in which A is given by, (200) and (212),

$$A = \frac{\sin(\phi \mp \delta)}{\cos \delta}.$$

the lower sign being for lower culmination. Solving for a we find

$$a = [\alpha - \tfrac{1}{2}(\theta_1 + \theta_2) - \Delta\theta + \tfrac{1}{2}(t_1 + t_2)] \frac{\cos \delta}{\sin(\varphi \mp \delta)}. \qquad (283)$$

a is expressed in time: in arc the azimuth correction is $15a$.

If a is $+$, the south end of the axis requires to be moved to the west; if $-$, to the east. This is readily done. Direct the telescope to a distant terrestrial object nearly in the horizon, make the movable micrometer wire vertical and set it on the object. Next move the wire through the distance a in the proper direction. This can be done when the value of one revolution of the screw is known (§ 60). Shift the whole instrument in azimuth by the proper screws until the micrometer wire is again on the object. The vertical plane of the polar axis should now coincide with the meridian.

If the value of a is large the observations should be repeated. If a is less than 3^s, it will cause no inconvenience and scarcely need be corrected.

Example. Wednesday, 1891 February 25. 51 *Cephei* was

observed at upper culmination with the 6-inch equatorial of the Detroit Observatory, as below. Determine the azimuth correction. The value of $\Delta\theta$ was $+14^m\ 36^s.0$.

Telescope.	Hour circle.	Chronometer.
West	$23^h\ 56^m\ 31^s$	$6^h\ 31^m\ 42^s$
East	24 0 42	6 35 29

The index correction of the hour circle was $-0^s.5$.

Naut. Alm., p. 303, α	$6^h\ 49^m\ 27^s.5$	δ	$+\ 87°\ 13'$
$\frac{1}{2}(\theta_1+\theta_2)$	6 33 35 .5	ϕ	$+\ 42\ \ 17$
$\Delta\theta$	+ 14 36 .0	$\cos\delta$	8.6863
$\frac{1}{2}(t_1+t_2)$	23 58 36 .0	$\sin(\phi-\delta)$	9.8490_n
	$-8\ .0$	$\dfrac{\cos\delta}{\sin(\phi-\delta)}$	0.069

The value of a was $-0.069\times-8^s.0=+0^s.6=+9''$; that is, the south end of the axis should be moved $9''$ west. This was too small to require correction.

136. *To adjust the declination verniers.* Direct the telescope to a star nearly in the zenith whose declination is known. Bring the star to the centre of the eye piece, using a high power, and clamp the instrument in declination. Set the verniers so that they read the star's declination. They will then be in adjustment.

137. *To centre the object glass.* Imperfect images are often due to the fact that the object glass is not properly centred. To test this adjustment, remove the eye piece and hold a candle flame in such a position that the images of it reflected from the inner surfaces of the object glass can be seen through the flame. If the object glass is perfectly centred all the images should coincide when the observer's eye and the centre of the flame are in the axis of the telescope. If they do not coincide raise one side of the object glass cell by the set screws until the coincidence is perfect.

CHRONOMETER CORRECTION.

138. The chronometer correction is quickly obtained, with an accuracy sufficient for all ordinary uses of the equatorial, by the following method:

Direct the telescope to a known star nearly in the zenith, note the chronometer time θ_1 when the star is on the point of intersection of the wires, and read the hour circle. Carry the telescope to the other side of the pier, observe the star as before at the time θ_2, and read the hour circle. Let the hour circle

readings corrected for index error be t_1 and t_2. We have, by (39),

$$a = \theta_1 + \Delta\theta - t_1,$$
$$a = \theta_2 + \Delta\theta - t_2,$$

neglecting only very small quantities and those eliminated by reversal. Therefore

$$\Delta\theta = a - \tfrac{1}{2}(\theta_1 + \theta_2) + \tfrac{1}{2}(t_1 + t_2). \qquad (284)$$

For many purposes an observation on one side of the pier will suffice; and we have

$$\Delta\theta = a + t_1 - \theta_1. \qquad (285)$$

Example. Ann Arbor, 1891 Feb. 25. The following observation of *Castor* was made with the 6-inch equatorial, to determine an approximate value of the chronometer correction.

Chronometer time,	θ_1	7^h	5^m	9^s
Hour circle,	t_1	23	52	3
Naut. Alm., p. 327,	a	7	27	39

Therefore, by (285), $\Delta\theta = +14^m\,33^s$.

139. *To direct the telescope to an object* whose right ascension (a) and declination (δ) are known, first determine whether the object is east or west of the meridian. If the right ascension is greater than the sidereal time it is east; if less, it is west. Generally, if the object is east, have the telescope west of the pier; if the object is west, have the telescope east of the pier. Move the telescope in declination till the declination circle reads δ. To the reading of the chronometer add the chronometer correction, and one or two minutes more for the time consumed in setting. Subtract a from this sum and set off the difference (which is the hour angle) on the hour circle. When the chronometer indicates the time used, the object should be seen in the telescope.

Conversely, if an unknown star is seen in the telescope, the chronometer time noted and the circle readings taken: then the declination circle reading is the star's declination; and the chronometer time of observation, plus the chronometer correction, minus the hour circle reading, is its right ascension.

These results are only approximate, of course, since the instrument will never be in perfect adjustment, and the star will not be seen in its true place, owing to the refraction, etc.

140. *Magnifying power.* The magnifying power of the telescope is equal to the focal length of the objective divided

by the focal length of the eye piece. It is therefore different for different eye pieces. The following method of determining it is simple and abundantly accurate.

Focus the telescope on a distant object, and direct it in the daytime to the bright sky. Hold a piece of thin, unglazed paper in front of the eye piece at such a distance that the bright disk formed on it is clearly defined. This disk is the minified image of the object glass. Measure its diameter by a finely divided scale held against the paper, and measure the diameter of the object glass. It can be shown that these diameters are to each other as the focal lengths of the eye piece and object glass. Their quotient is therefore the magnifying power. Thus, for the equatorial mentioned above, the diameter of the object glass is 6.05 inches, and the diameter of the bright disk for a certain eye piece is 0.08 inch. The magnifying power is, therefore, $6.05 \div 0.08 = 76$.

A definite statement of the magnifying power to be used in observing an object cannot be made. A higher power can be used when the *seeing* is good, *i. e.*, when the images in the telescope are steady and well defined, than when the seeing is poor. Lower powers must in general be used with nebulæ and comets. The very highest powers can be used with stars and some of the planetary nebulæ, if the seeing is good. Further than this, the observer must select that eye piece which on trial gives the best results.

141. *The field of view* is the circular portion of the sky which can be seen through the telescope at one time. Its diameter is equal to the angle contained by two rays drawn from the centre of the object glass to the two extremities of a diameter of the eye piece. The diameter is equal to the time required by an equatorial star to traverse it, which can be directly observed.

THE FILAR MICROMETER.

142. The micrometer as applied to the transit instrument or zenith telescope is described in § 59. The *filar micrometer*, applied to the equatorial, differs only in that it has a *position circle* perpendicular to the line of sight, and a device for moving the whole system of wires in their plane without changing their relative positions. Provision is also made for illuminating the wires. The value of a revolution of the screw is determined

best by observing a circumpolar star near culmination, § 60, (*b*).

143. *To determine the apparent place of an object.* Select a known* star, which is called a *comparison star*, whose right ascension and declination differ as little as possible from that of the object. Revolve the micrometer until the star in its diurnal motion follows along the micrometer wire. The transverse wire, which is parallel to the screw itself, should then coincide with a declination circle. Direct the telescope just in advance of the two objects. The diurnal motion will carry them across the field. Note the chronometer times when they cross the transverse wire or wires. The difference of these times is the difference of their right ascensions. Also, when the first or *preceding* object enters the field, move the whole system of wires until the object follows along the fixed wire parallel to the micrometer wire. When the second or *following* object reaches the middle of the field, bisect it with the micrometer wire. Read the micrometer in this position, and also in coincidence with the fixed wire. The difference of the readings multiplied by the value of a revolution of the screw is the difference of the declinations of the two objects.

In case the two sets of wires are not at right angles, an error is introduced. This should always be eliminated by revolving the micrometer 180° by means of the position circle, and making an equal number of observations in that position.

Many observers prefer to observe only one co-ordinate at a time. A good program is to measure the difference of declination, revolve the micrometer 90° and observe the difference of right ascension, then revolve 90° more and measure the difference of declination again.

In any case, the observations should be repeated several times, and the means of all the observations adopted. If the object has a proper motion, the differences in right ascension and declination are those corresponding to the instant when that object was observed: that is, the mean of the chronometer times plus the chronometer correction.

The mean place of the comparison star will be given for the epoch of the catalogue which contains it. Reducing this to the mean place for the beginning of the year of observation by § 46,

* That is, a star whose accurate position is given in one or more of the star catalogues. The star can be identified in the sky as well as in the catalogue by the methods § 139.

47 or 52, thence to the apparent place for the instant of observation by § 54, and applying the micrometer differences to the apparent place, we obtain the observed place of the object. This must be corrected for refraction and parallax.

The refraction correction will be small, since the star and object will be refracted nearly the same amount in nearly the same direction. An equatorial is a fixed part of an observatory, and tables of differential refractions in right ascension and declination in every part of the sky should be computed for the latitude of the observatory. The corrections could then be taken from the tables very quickly.

Until such tables are constructed the corrections can be computed by the following method: Let t_0 be the mean of the hour angles of the star and object, δ_0 the mean of their declinations, and z the mean of their zenith distances. Then if in (30) and (31) we put

$$n \sin N = \cos \varphi \cos t_0, \qquad n \cos N = \sin \varphi,$$

we can write

$$\tan N = \cot \phi \cos t_0, \ \tan q = \frac{\tan t_0 \sin N}{\cos(\delta_0 + N)}, \ \tan z = \frac{\cot(\delta_0 + N)}{\cos q} \tag{287}$$

Having found N and z from these, the corrections to the observed place will be given by *

$$\Delta\alpha = \frac{\kappa}{15} \cdot \frac{\delta' - \delta}{\sin^2(\delta_0 + N)} \cdot \frac{\tan t_0 \sin N \cos(2\delta_0 + N)}{\cos^2 \delta_0}, \tag{288}$$

$$\Delta\delta = \kappa \cdot \frac{\delta' - \delta}{\sin^2(\delta_0 + N)}, \tag{289}$$

in which $\delta' - \delta$ is the declination of the object minus the declination of the star expressed in seconds of arc, and κ is defined by

$$\kappa = \mu'' B T \gamma^{\lambda''}. \tag{290}$$

B, T and γ have the same significance as in § 30, and their values are given in Table I. The values of log μ'' and λ'' are tabulated below with the argument z.

z	log μ''	λ''	z	log μ''	λ''
0°	6.446	1.00	80°	6.395	1.10
45	6.444	1.00	82	6.370	1.15
60	6.440	1.01	83	6.351	1.18
70	6.433	1.03	84	6.323	1.21
75	6.422	1.05	85	6.285	1.24

*These equations are derived in *Chauvenet's Spherical and Practical Astronomy*.

It is only in the most refined measurements and in extreme states of the weather that the barometer and thermometer readings need be taken into account. With comets it will scarcely ever be necessary.

Likewise, tables should be constructed which give the parallax corrections in right ascension and declination for an object in any part of the sky, whose distance from the observer is equal to the sun's mean distance from the earth. Then for any object whose distance is Δ we would have only to divide the tabular parallax by Δ to obtain the desired correction.

If such tables are not at hand the corrections (except for the moon) can be computed from the following equations, which can be derived by methods analogous to those of §§ 28 and 29.

$$\left.\begin{aligned} \Delta\alpha &= \tfrac{1}{15}\cdot\frac{\pi\rho\cos\phi'}{\Delta}\cdot\frac{\sin t}{\cos\delta}, \\ \tan\gamma &= \frac{\tan\phi'}{\cos t}, \\ \Delta\delta &= \frac{\pi\rho\sin\phi'}{\Delta}\cdot\frac{\sin(\gamma-\delta)}{\sin\gamma}, \end{aligned}\right\} \qquad (291)$$

in which π is the sun's equatorial horizontal parallax, and Δ the object's distance expressed in terms of the sun's mean distance from the earth as the unit.

Four-place tables are sufficient for computing the refraction and parallax corrections.

Example. 1890 July 23, I made the following observation of Coggia's comet with the 12-inch equatorial of the Lick Observatory. Required its apparent place.

The comet was south of and preceding the 6th magnitude star No. 1518 *Pulcova Catalogue* for 1855.0. The reading of the position circle when the star followed along the micrometer wire was 201°.35. The micrometer readings when the micrometer and fixed wires coincided were

	19.947
	.947
	.947
Mean	19.947

When the comet was in the centre of the field the fixed wire was made to bisect it and the chronometer time was noted. When the star reached the centre of the field (nearly four minutes later) the micrometer wire was made to bisect it and the micrometer

reading noted. In this way the difference of the declinations was observed, as below.

Chronometer.	Micrometer.	Remarks.
$16^h\ 43^m\ 11^s$	22.954	Very windy.
48 59	23.822	" "
54 2	24.550	" "
Means 16 48 44	23.775	

The micrometer was rotated 90° until the circle read 291°.35, and the chronometer times of transit over the two wires noted, as below.

Comet.		Star.		Difference.
$17^h\ 0^m\ 51^s.7$	$0^m\ 59^s.2$	$17^h\ 4^m\ 44^s.6$	$4^m\ 51^s.8$	$-3^m\ 52^s.75$
5 42 .3	5 49 .5	9 34 .0	9 41 .3	3 51 .75
10 37 .8*	10 46 .1	14 28 .0	14 36 .3	3 50 .20
23 3 .5	23 11 .5	26 49 .8	26 58 .1	3 46 .45
27 39 .5*	27 52 .2	31 24 .6	31 37 .2	—3 45 .05
Means $17^h\ 13^m\ 39^s$				—3 49 .24

The micrometer was rotated 90° further until it read 21°.35. and the difference of the declinations measured again, as below.

Chronometer.	Micrometer.	Remarks.
$17^h\ 34^m\ 22^s$	9.751	Very windy.
39 44	8.867	" "
48 40	7.545	" "
Means 17 40 55	8.721	

Readings for coincidence of wires,

19.944
.946
.943
Mean 19.944

The value of one revolution of the screw is $R = 14''.058$. We shall combine the two differences of declination, thus:

Chronometer.	Diff. of decl.
$16^h\ 48^m\ 44^s$	$-3.828\ R$
17 40 55	$-11.223\ R$
Means 17 14 49	$-7.525\ R = -1'\ 45''.8$

The chronometer time for the declinations is $1^m\ 10^s$ greater than that for the right ascensions. From the two measured declinations it is found that the declination changed $2''.3$ in 1^m 10^s. Therefore at $17^h\ 13^m\ 39^s$ the difference of declination was $-1'\ 43''.5$.

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* The distance between the wires was changed intentionally.

The mean place of the comparison star for 1855.0 given by the catalogue was

$$\alpha = 9^h\, 29^m\, 17^s.57\,, \qquad \delta = +40°\, 53'\, 16''.1\,.$$

The mean place for 1890.0 was, by § 47,

$$\alpha = 9^h\, 31^m\, 29^s.56\,, \qquad \delta = +40°\, 43'\, 58''.8\,;$$

and the apparent place for sidereal time 1890 July 23^d 17^h, by § 54,

$$\alpha' = 9^h\, 31^m\, 28^s.90\,, \qquad \delta' = +40°\, 44'\, 4''.2\,.$$

Therefore the observed place of the comet at $17^h\, 13^m\, 39^s$ was

$$\alpha = 9^h\, 27^m\, 39^s.66\,, \qquad \delta = +40°\, 42'\, 20''.7\,.$$

The chronometer correction was $-1^m\, 19^s$. We have

Chronometer time,	θ'	$17^h\, 13^m\, 39^s$
Chronometer corr.,	$\Delta\theta$	— 1 19
Sidereal time,	θ	17 12 20
Right ascension,	α	9 27 40
Hour angle,	t	7 44 40

The corrections for differential refraction corresponding to this hour angle and declination, taken from the tables constructed for the Lick Observatory, are

$$\Delta\alpha = -0^s.04\,, \qquad \Delta\delta = -0''.6\,.$$

The corrections for parallax at the unit distance (the sun's mean distance) are

$$\Delta\alpha = +0^s.56\,, \qquad \Delta\delta = +6''.1\,.$$

The comet's distance was 1.57, and therefore the required corrections for parallax are

$$\Delta\alpha = +0^s.36\,, \qquad \Delta\delta = +3''.8\,.$$

Applying these corrections to the observed place we obtain the following apparent place of the comet:

Mt. Hamilton sid. time.	Apparent α	Apparent δ
1890 July 23^d $17^h\, 13^m\, 39^s$	$9^h\, 27^m\, 39^s.98$	$+40°\, 42'\, 23''.9$

144. *To find the position angle and distance of two stars.** Revolve the micrometer until one of the stars by its diurnal motion follows along the transverse wire, and take the reading P_0 of the position circle. The reading when the transverse wire is in a declination circle is $P_0 \pm 90°$. Direct the telescope so that the two stars are on opposite sides of the centre of the field at nearly equal distances from it, and keep it so directed by means of the driving clock. Revolve the micrometer until the

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* The *position angle* is the angle which the line joining the two objects makes with the hour circle through one of them, reckoned from the north toward the east through 360°. The *distance* is the length of the arc of the great circle joining them.

transverse wire passes through the stars and take the reading P of the circle. The required position angle p is given by

$$p = P - (P_0 \pm 90^\circ). \tag{292}$$

To measure the distance between the stars bisect one of them with the fixed wire and the other with the micrometer wire. Let m_0 be the micrometer reading when the two wires coincide, m the reading when the micrometer wire bisects the star, and R the value of one revolution of the screw. Then the distance s between the stars is (very nearly)

$$s = \tfrac{1}{2}(m - m_0)R. \tag{293}$$

In very accurate measurements bisect the two stars as above and take the reading m. Move the micrometer wire to the other side of the fixed wire, bisect the stars with the wires in that order, and take the reading m'. The distance is now given by

$$s = \tfrac{1}{2}(m' - m)R. \tag{294}$$

In this way several systematic and personal errors are eliminated. This method is called *the method of double distances.*

The measured angle p is the angle between the arc joining the stars and the hour circle through the middle point of that arc. The values of s and p are also affected by refraction. In the case of double stars it will seldom be necessary to apply any correction for refraction.

When the value of s is large the observation is generally for the purpose of determining the position of an object by comparing its place with that of a known star. If the declination is less than 80° the following method of reducing the observations is sufficiently accurate. Let α, δ be the co-ordinates of the star, and α', δ' those of the object. Projecting the arc s upon the hour circle through the star, and also upon the small circle through the star perpendicular to the hour circle, we have (very nearly)

$$\delta' - \delta = s \cos p, \tag{295}$$

$$\alpha' - \alpha = s \sin p \sec \tfrac{1}{2}(\delta' + \delta). \tag{296}$$

Applying these differences to the apparent place of the star, and correcting for refraction and parallax as in § 143, the result is the apparent place of the object.

If the object is within 10° of the pole the arc s cannot be projected in right ascension and declination without introducing an error that is sometimes larger than the probable error of observation. If p and s are the observed quantities, and δ, the

mean of the declinations of the stars (which can be found with sufficient accuracy from (295),) the equivalent differences in right ascension and declination can be found from the following equations: *

$$\left.\begin{aligned} p_0 &= p + \tfrac{1}{16}\, s^2 \sin 1'' \sin 2p\,(1 + 2\tan^2\delta_0), \\ \Delta p &= \tfrac{1}{2} s \sin p \tan \delta_0\,, \\ \tan \tfrac{1}{2}(\delta' - \delta) &= \frac{\tan \frac{1}{2} s \cos p_0}{\cos \Delta p}, \\ \sin \tfrac{1}{2}(\alpha' - \alpha) &= \frac{\sin \frac{1}{2} s \sin p_0}{\cos \frac{1}{2}(\delta' + \delta)}. \end{aligned}\right\} \qquad (297)$$

The position angle and distance should be observed four or five times in quick succession, and the means of the results adopted as the values of p and s at the mean of the times of observation. But this method will not be sufficiently accurate if the object has a large proper motion (§ 48). In this case it is best to compute the differences of right ascension and declination for each observation, and take the mean of the results.

THE RING MICROMETER.

145. This consists of a narrow metal ring, one or both of its edges turned exactly circular, attached to a thin piece of glass in the focus of an eyepiece. When the eyepiece is put on the telescope and focused, the ring is in the focus of the object glass, and its plane is perpendicular to the optical axis.

If the times of transit of two stars over the edges of the ring are observed, the differences of their right ascensions and declinations can be found. But results obtained in this way can be regarded only as approximately correct, and the ring micrometer should never be used unless, in case of great haste, there is not time to attach the filar micrometer and adjust its wires by the diurnal motion. †

146. *To find the radius of the ring.* Select two stars whose declinations are accurately known, the difference of whose declinations is a little less than the diameter of the ring, and whose right ascensions do not differ more than three or four minutes.‡ When these stars are nearly on the observer's meridian observe their transits over the edge of the ring.

* These equations are derived in *Chauvenet's Spherical and Practical Astronomy.* Lack of space prevents their derivation here.

† The principal advantage of the ring micrometer is that it can be used with an instrument not mounted as an equatorial.

‡ Two stars in the *Pleiades* can always be found to fulfill these conditions.

In Fig. 16 let $C\,D\,D'\,C'$ represent the ring; $C\,D$ the path of one star (α, δ), and t_1 and t_2 the observed sidereal times of its transit over C and D; $C'\,D'$ the path of the other star (α', δ'), and t_1' and t_2' the times of its transit over C' and D'. Draw $M\,M'$ perpendicular to $C\,D$ and $C'\,D'$. Draw the radii $C\,O$ and $C'\,O$, and let r represent their value in seconds of arc. If we put

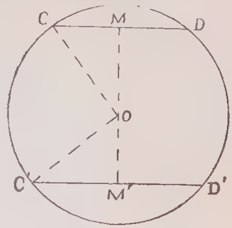

Fig. 16.

$$C\,O\,M = \gamma, \qquad C'\,O\,M' = \gamma',$$

we can write

$$O\,M \;= r\cos\gamma, \qquad C\,M \;= r\sin\gamma,$$
$$O\,M' = r\cos\gamma', \qquad C'\,M' = r\sin\gamma';$$

and therefore

$$M\,M' = r(\cos\gamma' + \cos\gamma) = 2\,r\;\cos\tfrac{1}{2}(\gamma' + \gamma)\cos\tfrac{1}{2}(\gamma' - \gamma), \tag{298}$$
$$C'\,M' + C\,M = r(\sin\gamma' + \sin\gamma) = 2\,r\sin\;\tfrac{1}{2}(\gamma' + \gamma)\cos\tfrac{1}{2}(\gamma' - \gamma), \tag{299}$$
$$C'\,M' - C\,M = r(\sin\gamma' - \sin\gamma) = 2\,r\cos\tfrac{1}{2}(\gamma' + \gamma)\sin\tfrac{1}{2}(\gamma' - \gamma). \tag{300}$$

We have

$$M\,M' = \delta' - \delta, \quad C\,M = \tfrac{15}{2}(t_2 - t_1)\cos\delta, \quad C'\,M' = \tfrac{15}{2}(t_2' - t_1')\cos\delta';$$

and if we put

$$\tfrac{1}{2}(\gamma' + \gamma) = A, \qquad \tfrac{1}{2}(\gamma' - \gamma) = B,$$

we can write

$$\tan A = \frac{C'\,M' + C\,M}{M\,M'} = \frac{\tfrac{15}{2}(t_2' - t_1')\cos\delta' + \tfrac{15}{2}(t_2 - t_1)\cos\delta}{\delta' - \delta}, \tag{301}$$

$$\tan B = \frac{C'\,M' - C\,M}{M\,M'} = \frac{\tfrac{15}{2}(t_2' - t_1')\cos\delta' - \tfrac{15}{2}(t_2 - t_1)\cos\delta}{\delta' - \delta}, \tag{302}$$

$$r = \frac{M\,M'}{2\cos A\cos B} = \frac{\delta' - \delta}{2\cos A\cos B}. \tag{303}$$

The distance between the stars is affected by refraction) Since the observations are made near the meridian, the refraction in right ascension can be neglected, and it will be sufficiently accurate to consider that its effect upon the difference of the declinations is equal to the difference of the refraction in zenith distance of the stars when they are on the meridian. If we let $r' - r''$ be the difference of the refractions, $z' - z''$ the difference of the zenith distances in minutes of arc, and z the mean zenith distance of the two, the difference of the refractions is given by (256), or it can be taken from the table in § 112 based upon (256). The difference of the declinations furnished by the star catalogues must be numerically decreased by the difference of the refractions before substituting in the above equations.

The observations should be repeated several times and the mean of all the results adopted as the value of r.

Example. 1889 Jan. 25. The following times of transit of 23 *Tauri* and 27 *Tauri* over the outer edge of the ring micrometer of the $12\frac{1}{4}$-inch equatorial of the Detroit Observatory were noted, to determine the radius of the ring.

23 *Tauri*.	27 *Tauri*.
$t_1 = 3^h\, 30^m\, 41^s.5,$	$t_1' = 3^h\, 33^m\, 33^s.5,$
$t_2 = \quad 30 \;\; 53\,.0,$	$t_2' = \quad 33 \;\; 41\,.0.$

The mean places of these stars for 1850.0 are given in *Newcomb's Standard Stars.* Reducing them to the mean place for 1889.0 by § 52, and thence to the apparent place at the instant of observation by § 54, (b), we obtain

$$\delta = 23^\circ\, 36'\, 4''.13, \qquad \delta' = 23^\circ\, 42'\, 45''.40.$$

The zenith distance of the stars is about 18° and the difference of their zenith distances is about 7′. Entering the table in § 112 with $z' - z'' = 7'$ and $z = 18°$, one half the difference of the refractions is 0″.07, or the whole difference is 0″.14. Therefore the apparent difference of declination of the stars is

$$\delta' - \delta = 401''.13.$$

From (301) and (302) we find

$$A = 180^\circ\, 1'\, 33'', \qquad B = 3^\circ\, 55'\, 37'';$$

and then, from (303),

$$r = 211''.4.$$

The mean of nine results gave

$$r = 210''.7 \pm 0''.18.$$

147. *To determine the difference of the right ascensions and declinations of two stars.* Observe the transits over the edge of the ring, as in § 146. Using the notation of § 146, the difference of the right ascensions is given by

$$\alpha' - \alpha = \tfrac{1}{2}(t_1' + t_2') - \tfrac{1}{2}(t_1 + t_2). \tag{304}$$

Letting $OM = d$, and $OM' = d'$ we can write

$$\sin\gamma = \frac{\frac{15}{2}\cos\delta}{r}(t_2 - t_1), \qquad \sin\gamma' = \frac{\frac{15}{2}\cos\delta'}{r}(t_2' - t_1'), \tag{305}$$

$$d = r\cos\gamma, \qquad d' = r\cos\gamma'. \tag{306}$$

The difference of the declinations is given by

$$\delta' - \delta = d' \pm d. \tag{307}$$

The lower sign is used if the stars are observed on the same side of the centre of the ring. Equations (306) do not determine the signs of d and d', but there will never be any ambiguity if

the observer notes the positions of the two stars with reference to the centre of the ring.

Differences obtained in this way are slightly in error on account of refraction; on account of the fact that the paths of the stars are arcs and not chords of the circle (except for equatorial stars); and on account of the proper motion of one of the stars (if it have a proper motion). But as stated above, the ring micrometer should not be used on an equatorial telescope when exact measurements are required; so that the corrections for these errors will seldom be justified, and we shall not consider them here.

Example. 1888 Sept. 8. The position of *Comet c* 1888 was compared with that of the star 13^h, 1197 in *Weisse's Bessel's Catalogue* by observing the transits of the star and comet over the inner edge of the ring micrometer of the $12\frac{1}{4}$-inch equatorial of the Detroit Observatory. The times of transit were

Star.	Comet.
$t_1 = 19^h\,18^m\,34^s.1,$	$t_1' = 19^h\,19^m\,12^s.6,$
$t_2 = \;18\;\;50.0,$	$t_2' = \;19\;\;31.1.$

The image of the star was north of the centre and that of the comet was south. The radius of the ring is $171''.6$.

The apparent place of the star was

$$\alpha = 13^h\,56^m\,3^s.06, \qquad \delta = +31°\,13'\,49''.6.$$

The declination of the comet was approximately $+31°\,18'$.

Substituting in (304) we find

$$\alpha' - \alpha = +0^m\,39^s.80.$$

The solution of (305) and (306) is here given.

$\log \frac{15}{2}$	0.87506	$\log \frac{15}{2}$	0.87506
$\cos \delta$	9.93201	$\cos \delta'$	9.93169
a.c. $\log r$	7.76548	a.c. $\log r$	7.76548
$\log(t_2 - t_1)$	1.20140	$\log(t_2' - t_1')$	1.26717
$\sin \gamma$	9.77395	$\sin \gamma'$	9.83940
$\cos \gamma$	9.90542	$\cos \gamma'$	9.85912
$\log r$	2.23452	$\log r$	2.23452
d	$138''.0$	d'	$124''.1$

$$\delta' - \delta = d' + d = +262''.1 = +4'\,22''.1$$

These approximate values of the apparent differences $\alpha' - \alpha$ and $\delta' - \delta$ correspond to the Ann Arbor sidereal time 1888 Sept. $8^d\,19^h\,19^m\,22^s$.

APPENDIXES.

A. HINTS ON COMPUTING.

The numerical calculations required in the problems of practical astronomy are generally a source of discouragement to the beginner, even though he be a skillful mathematician. Practice in making extensive series of computations, however, very soon suggests to him various devices for avoiding much of the labor and saving the greater part of the time. Every computer acquires methods peculiarly his own; yet the following hints will possibly be useful to many.

Only those logarithmic tables should be employed which contain the auxiliary tables of proportional parts on the margins of the pages, excepting possibly three-place and four-place tables. They enable the computer to make nearly all the interpolations mentally; and the use of any other tables, for any purpose whatever, cannot be too strongly condemned.

The following are recommended:

Bruhns's or *Vega's* 7-place tables.

Bremiker's 6-place tables.

Hussey's, or *Newcomb*'s, or *Becker*'s 5-place tables.

Zech's addition and subtraction logarithmic tables. *

Barlow's tables of squares, square-roots, etc.

Crelle's multiplication and division tables.

If extensive computations are made with 7-place tables and the interpolations carried to hundredths of seconds, the results are usually accurate within a tenth of a second. If 6-place tables are used and the interpolations carried to tenths of a second, the results are usually accurate within a second. If 5-place tables are employed and the interpolations carried to seconds or to hundredths of minutes, the results are usually accurate within five seconds.

First of all, an outline of the *whole solution* should be prepared by writing in a vertical column the symbols of all the functions that will be used. Those should be placed adjacent

* *Bremiker's* tables contain *Gauss's add.* and *sub* log. tables to six places. *Hussey's* and *Becker's* contain *Zech's* to five places, and *Newcomb's* contain *Gauss's* to five places.

to each other which are to be combined, as shown by the formulae. If a number of similar solutions for different values of the variable or variables are to be made, a vertical column should be arranged for each on the right of the column of functions, which thus serves for all and the computations in the several columns should be carried on *simultaneously*. If the solutions are made for equidistant values of the variable or variables, this method affords a valuable check on the accuracy of the results; for all the quantities which are in the same horizontal line should differ systematically from each other as we go from the first column to the last. By subtracting the result in each column from the corresponding result in the next column to the right, any error will be detected very quickly by the fact that the *differences* will not vary properly. This method is called the *method of differences*. It will not detect systematic errors: that is, errors affecting all the columns alike.

If the sine, cosine, tangent, etc., of the same angle are required, they should all be taken from the tables at one opening. *Avoid turning twice to the same angle in the same solution.* The interpolations can be checked by subtracting mentally the last two figures of the cosine from those of the sine and comparing the result with the last two places of the tangent; and similarly in other cases.

The tangent of an angle always varies more rapidly than its sine or cosine, and for this reason the value of an angle should be taken from the tables by means of its tangent if great accuracy is required.

Many of the operations can be performed mentally, thereby saving much time. Thus, two numbers can be added or subtracted mentally *from left to right*, or a number multiplied or divided by two from left to right, and the result held in mind while we turn to the tables and take out the proper angle or function. This has been done very largely in the solutions of the examples in this book. Just how far the student should carry the method depends upon the individual. The beginner will find it perplexing and a fruitful source of error, but after some practice he can perform the operations quickly and accurately. It should be said that many experienced computers prefer to set down the results in the usual way.

If there are several factors in the numerator or denominator of an expression to be evaluated, do not add the logarithms in

the numerator together, those of the denominator together, and take the difference; but form the arithmetical complement of each logarithm in the denominator *mentally* by subtracting it from 10, from left to right, and setting down the result in the proper place. All the factors can then be combined by one addition.

When a constant quantity is to be used several times, it should be written on the margin of a slip of paper and held over the quantities with which it is to be combined.

If two quantities are given by their logarithms, and the logarithm of their sum or difference is required, it should be found by means of addition and subtraction logarithmic tables. The result will be obtained more quickly and accurately than by means of the ordinary tables.

Whenever the formulae furnish checks on the accuracy of the solution, they should generally be applied. The experienced computer usually detects an error very quickly.

If the trigonometrical function or other logarithmic function is negative, write the subscript $_n$ after it.

Do not use negative characteristics. Increase them by 10 if they are naturally negative.

If two quantities are to be combined which are separated by one or more lines, hold a pencil or slip of paper over the intervening quantities and the two can then be combined as conveniently as if they were adjacent.

The example in § 13 will illustrate many of these methods. First write down the two columns of functions, as the outline of the solution of (12), (13), (14) and the check equation (9) The values of φ, z and A are inserted. From the tables tan z and sin z are found and written opposite their symbols; and likewise cos A, tan A and sin A. The sum of tan z and cos A is tan M. Add them mentally, enter the tables and take out M. Take out sin M at once. Subtract M from φ, mentally, find sec $(\varphi - M)$ and tan $(\varphi - M)$. The value of sec $(\varphi - M)$ is $10 - \cos(\varphi - M)$. Add the three logarithms to find tan t. Determine t from the tables and take cos t and cosec t out. The sum of tan $(\varphi - M)$ and cos t is tan δ. Add them mentally and take δ and sec δ from the tables. The sum of the last four logarithms is log 1. It should not differ more than one or two units of the last place from zero or ten.

The printed solution contains every figure that need be writ-

ten down. But possibly the beginner should write down tan M, $\varphi - M$, and tan δ.

B. COMBINATION AND COMPARISON OF OBSERVATIONS.

Formulæ resulting from the Method of Least Squares.

1. Direct observations of a quantity: n separate results, $m_1, m_2, \ldots m_n$ of equal weight.

Most probable value of quantity, $z = \frac{[m]}{n}$. *

Residuals, $z - m_1 = v_1,\ z - m_2 = v_2, \ldots z - m_n = v_n$.

Probable error of z, $r_0 = \pm 0.6745 \sqrt{\frac{[vv]}{n(n-1)}}$.

Probable error of a single observation, $r = \pm 0.6745 \sqrt{\frac{[vv]}{n-1}}$.

2. Direct observations of a quantity: n separate results, m_1, $m_2, \ldots m_n$ of unequal weights, $p_1, p_2, \ldots p_n$.

Most probable value of quantity, $z = \frac{[pm]}{[p]}$.

Probable error of z, $r_0 = \pm 0.6745 \sqrt{\frac{[pvv]}{[p](n-1)}}$.

Probable error of an obs'n of weight unity, $r = \pm 0.6745 \sqrt{\frac{[pvv]}{n-1}}$.

Weight of z, $P = [p]$.

Relation of weights to probable errors, $p_1 : p_2 : \ldots :: \frac{1}{r_1^2} : \frac{1}{r_2^2} : \ldots$

3. If $Z = az_1 \pm bz_2 \pm \ldots kz_n$, and the probable errors and weights of $z_1, z_2, \ldots z_n$ are $r_1, r_2, \ldots r_n$ and $p_1, p_2, \ldots p_n$, then the probable error and weight of Z are given by

$$r = \pm \sqrt{(ar_1)^2 + (br_2)^2 + \ldots (kr_n)^2}.$$

$$\frac{1}{p} = \frac{a^2}{p_1} + \frac{b^2}{p_2} + \ldots \frac{k^2}{p_n}.$$

4. In general, if $Z = f(z_1, z_2, \ldots z_n)$, the probable error of Z is

$$r = \pm \sqrt{\left(\frac{df}{dz_1}\right)^2 r_1^2 + \left(\frac{df}{dz_2}\right)^2 r_2^2 + \ldots + \left(\frac{df}{dz_n}\right)^2 r_n^2}.$$

5. Direct observations of a function of a quantity z: the separate results, $m_1, m_2, \ldots m_n$ of equal weight, and the form of the function, az. The observation equations are

$$\begin{aligned} a_1 z + m_1 &= 0, \\ a_2 z + m_2 &= 0, \\ &\cdots\cdots \\ a_n z + m_n &= 0. \end{aligned}$$

* The symbols [ ] signify the sum of all similar quantities. Thus,

$$[m] \equiv m_1 + m_2 + \ldots + m_n.$$
$$[pvv] \equiv p_1 v_1^2 + p_2 v_2^2 + \ldots + p_n v_n^2.$$

The most probable value of z and its probable error are

$$z = -\frac{[am]}{[aa]} \qquad r = \pm 0.6745 \sqrt{\frac{[vv]}{[aa]\ (n-1)}}.$$

If the observations are of unequal weights, multiply the observation equations through by the square roots of their respective weights, and proceed as before.

6. Direct observations of a function of two quantities, w and z: the separate results, $m_1, m_2, \ldots m_n$ of equal weights, and the form of the function, $aw + bz$. The observation equations are

$$\begin{aligned} a_1 w + b_1 z + m_1 &= 0, \\ a_2 w + b_2 z + m_2 &= 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ &\cdot\ \cdot\ \cdot \\ a_n w + b_n z + m_n &= 0. \end{aligned}$$

The normal equations are

$$\begin{aligned} [aa]w + [ab]z + [am] &= 0, \\ [ab]w + [bb]z + [bm] &= 0. \end{aligned}$$

Let

$$[bb] - \frac{[ab]}{[aa]}[ab] = [bb.1], \qquad [bm] - \frac{[ab]}{[aa]}[am] = [bm.1]$$

Then the most probable values of w and z are given by

$$z = -\frac{[bm.1]}{[bb.1]},$$

$$w = -\frac{[ab]}{[aa]}z - \frac{[am]}{[aa]}.$$

The weights of w and z are

$$p_z = [bb.1], \qquad p_w = \frac{[bb.1]}{[bb]}[aa].$$

The probable error of a single observation (of weight unity) is

$$r = \pm 0.6745 \sqrt{\frac{[vv]}{[n-2]}};$$

and the probable errors of w and z are

$$r_w = \frac{r}{\sqrt{p_w}}, \qquad r_z = \frac{r}{\sqrt{p_z}}.$$

If the observations are of unequal weights, multiply the observation equations through by the square roots of their respective weights and proceed as before.

7. Direct observations of a function of three quantities, x, y and z: the separate results, $m_1, m_2, \ldots m_n$ of equal weight, and the form of the function, $ax + by + cz$. The observation equations are

$$\begin{aligned} a_1 x + b_1 y + c_1 z + m_1 &= 0, \\ a_2 x + b_2 y + c_2 z + m_2 &= 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ &\cdot\ \cdot\ \cdot \\ a_n x + b_n y + c_n z + m_n &= 0. \end{aligned}$$

The normal equations are

$$
\begin{aligned}
[aa]x + [ab]y + [ac]z + [am] &= 0,\\
[ab]x + [bb]y + [bc]z + [bm] &= 0,\\
[ac]x + [bc]y + [cc]z + [cm] &= 0.
\end{aligned}
$$

Let

$$
\begin{aligned}
&[bb] - \frac{[ab]}{[aa]}[ab] = [bb.1], \quad [bc] - \frac{[ab]}{[aa]}[ac] = [bc.1],\\
&[bm] - \frac{[ab]}{[aa]}[am] = [bm.1],\\
&[cc] - \frac{[ac]}{[aa]}[ac] = [cc.1], \quad [cm] - \frac{[ac]}{[aa]}[am] = [cm.1],\\
&[cc.1] - \frac{[bc.1]}{[bb.1]}[bc.1] = [cc.2], \quad [cm.1] - \frac{[bc.1]}{[bb.1]}[bm.1] = [cm.2].
\end{aligned}
$$

Then the most probable values of x, y and z are given by

$$
\begin{aligned}
z &= -\frac{[cm.2]}{[cc.2]},\\
w &= -\frac{[bc.1]}{[bb.1]}z - \frac{[bm.1]}{[bb.1]},\\
x &= -\frac{[ab]}{[aa]}w - \frac{[ac]}{[aa]} - \frac{[am]}{[aa]}.
\end{aligned}
$$

The weights of x, y and z are given by

$$
\begin{aligned}
p_z &= [cc.2],\\
p_w &= \frac{[cc.2]}{[cc.1]}[bb.1].\\
p_x &= \frac{[cc.2]}{[cc.1]_a} \cdot \frac{[bb.1]}{[bb]}[aa],
\end{aligned}
$$

in which

$$
[cc.1]_a = [cc] - \frac{[bc]}{[bb]}[bc].
$$

The probable error of a single observation (of weight unity) is

$$
r = \pm 0.6745\sqrt{\frac{[vv]}{n-3}},
$$

and the probable errors of x, y and z are

$$
r_x = \frac{r}{\sqrt{p_x}}, \quad r_y = \frac{r}{\sqrt{p_y}}, \quad r_z = \frac{r}{\sqrt{p_z}}
$$

If the observations are of unequal weights multiply the observation equations through by the square roots of their respective weights, and proceed as before.

C. OBJECTS FOR THE TELESCOPE.

Besides the moon, the planets and the Milky Way, the objects in the following list will be of interest to the student. Fuller descriptions of them, with many valuable hints on the

use of the telescope, can be found in *Webb's Celestial Objects for the Common Telescope*, which is an excellent guide for the observer. Every student should provide himself with a good star atlas, which is an indispensable companion. *Klein's Star Atlas*, or *Proctor's Star Atlas*, or *Heis's Atlas Coelestis* is recommended.

α, 1900.0	δ, 1900.0	Object: description: remarks.
$0^h\ 34^m.8$	$+55°\ 59'$	α *Cassiopeiæ*, variable, $2^m.2$ to $2^m.8$, period irregular but about 79^d.
0 37 .3	+40 43	*The Great Nebula in Andromeda.* One of the most interesting in the sky, large, $2\frac{1}{2}°$ by 4°, easily visible to the naked eye. Its constitution is still somewhat uncertain. A small companion nebula lies 22′ south.
0 53 .4	+81 20	*U Cephei*, variable, $7^m.1$ to $9^m.2$, period $2^d.5$.
1 18 .9	+67 36	ψ *Cassiopeiæ*, triple, $A\ 4^m.5$, $B\ 9^m$, $C\ 10^m$. $AB = 30''$, $BC = 3''$.
1 22	+88 46	α *Ursæ Minoris* or *Polaris*, the standard 2^m star, a 9^m companion at $s = 18''.5$.
1 48 .0	+18 48	γ *Arietis*, double, $4^m.5$ and 5^m, $p = 179°$, $s = 8''$.
1 57 .7	+41 51	γ *Andromedæ*, double, one yellow $3^m.5$ and one green $5^m.5$ $p = 63°$, $s = 10''$. The $5^m.5$ is also double, but close and difficult.
2 12 .0	+56 41	*Cluster in Perseus.* A magnificent object with a low power. Another fine cluster 3^m east.
2 14 .3	− 3 26	*o Ceti.* interesting variable, irregular, $1^m.7$ to $9^m.5$, period about 331^d.
3 1 .7	+40 34	β *Persei* (*Algol*), interesting variable, $2^m.3$ to $3^m.5$, period $2^d\ 20^h\ 48^m\ 55^s$.
3 40 .2	+23 27	*Nebula in the Pleiades*, very faint and difficult, *Merope* in its north extremity.
4 7 .6	+50 59	*Cluster in Perseus*, good with low power.
4 9 .6	−13 0	*Planetary nebula in Eridanus*, circular, 12^m star in center. Probably stellar.
4 30 .2	+16 19	α *Tauri* (*Aldebaran*), 1^m star, red.
5 9 .3	+45 54	α *Aurigæ* (*Capella*), 1^m star, white.
5 9 .7	− 8 19	β *Orionis* (*Rigel*), double, 1^m and 9^m, $s = 9''.5$.
5 28 .5	+21 57	*Nebula in Taurus* large, faint, oblong.
5 30 .4	− 5 27	*The Great Nebula in Orion*, the most interesting nebula visible in this latitude, about 3° by 5° in size. Near its densest part is the quadruple star θ *Orionis*, which forms the *Trapezium*. Its spectrum indicates a gaseous composition
5 35 .7	− 2 0	ζ *Orionis* triple, $A\ 3^m$, $B\ 6^m.5$, $C\ 10^m$, $AB = 2''.5$, $AC = 57''$.
5 49 .8	+ 7 23	α *Orionis* (*Betelgeux*), 1^m star, red.
6 2 .7	+24 21	*Cluster in Gemini.* fine field with low power.
6 37 .4	+59 33	12 *Lyncis*, triple, $A\ 6^m$, $B\ 6^m.5$, $C\ 7^m.5$, $AB = 1''.5$, $AC = 9''$.
6 40 .7	−16 34	α *Canis Majoris* (*Sirius*), the brightest star in the sky. A close 10 mag. companion is difficult in most powerful telescopes.

α, 1900.0	δ, 1900.0	Object: description: remarks.
$7^h\ 14^m.1$	$+2°\ 210'$	*δ Geminorum*, double, one yellow $3^m.5$, the other red 8^m. $p=205°$, $s=7''$.
7 28 .2	+32 6	*α Geminorum* (*Castor*), fine double, 3^m and $3^m.5$, $s=5''.5$.
7 34 .1	+ 5 29	*α Canis Minoris* (*Procyon*) 1^m star, yellow.
8 34 .5	+20 17	*Cluster in Cancer* (*Praesepe*), fine field with low power.
8 45 .7	+12 10	*Cluster in Cancer*, about 200 stars, 9^m to 15^m.
9 47 .2	+69 36	*Nebulæ in Ursa Major*, two nebulæ 30′ apart, preceding one brighter with bright nucleus.
10 14 .4	+20 21	*γ Leonis*, fine double, one yellow 2^m, the other rather green, $3^m.5$, $s=3''$.
10 19 .9	—17 39	*Planetary Nebula in Hydra*, fairly bright.
11 12 .5	+59 19	*Nebula in Ursa Major*, small, bright, with nucleus.
11 47 .7	+37 33	*Nebula in Ursa Major*, bright, 3′ to 4′ in diam.
12 5 .0	+19 6	*Cluster in Coma Berenices*, globular, bright, well resolved in large telescope.
12 34 .8	—11 4	*Nebula in Virgo*, elliptical, 30″ by 5′, fine field with low power.
12 36 .6	— 0 54	*γ Virginis*, double, 4^m and 4^m, period about 170 years.
12 51 .4	+38 51	*α Canum Venaticorum*, fine double, $2^m.5$ and $6^m.5$, $p=228°$, $s=20''$.
13 19 .9	+55 27	*ζ Ursæ Majoris*, fine double, 3^m and 5^m, $s=14''$.
13 37 .5	+28 52	*Cluster in Canes Venatici*, bright, globular, probably more than 1,000 stars.
14 11 .1	+19 43	*α Boötis* (*Arcturus*), 1^m star, yellow.
14 40 .6	+27 30	*ϵ Boötis*, beautiful double, 3^m yellow and 7^m blue, $s=3''$, $p=328°$.
15 14 .1	+32 1	*U Coronæ*, variable, $7^m.5$ to 8.9, period $3^d\ 10^h\ 51^m$.
16 23 .3	—26 13	*α Scorpii*, double, 1^m red and 7^m green or blue, $s=3''$.
16 37 .5	—31 47	*ζ Herculis*, binary, 3^m and 6^m, $s=1''.5$, period about 35 years.
16 38 .1	+36 39	*The Cluster in Hercules*, globular, finest of its kind, containing several thousand stars.
16 40 .3	+23 59	*Nebula in Hercules*, planetary, 8″ in diameter.
17 10 .1	+14 30	*α Hercules*, variable, $3^m.1$ to $3^m.9$, irregular period; companion $5^m.5$ at $p=116°$, $s=4''.7$.
17 11 .5	+ 1 19	*U Ophiuchi*, $6^m.0$ to $6^m.7$, period $20^h\ 8^m$.
17 51 .1	− 18 59	*Cluster in Ophiuchus*, good field with low power.
17 58 .6	+66 38	*Nebula in Draco*, planetary, bright, diameter 35″, very near pole of ecliptic.
18 7 .3	+ 6 50	*Nebula in Ophiuchus*, planetary, bright, diameter 5″.
18 33 .6	+38 41	*α Lyræ* (*Vega*), 1^m star.
18 41 .0	+39 34	*ϵ Lyræ*, a multiple star, $A\ 5^m$, $B\ 6^m.5$, $C\ 5^m$, $D\ 5^m.5$, $AB=3''$, $CD\ 2''.5$, $AC\ 207''$. Three small stars between AB and CD.
18 46 .4	+33 15	*β Lyrae* variable, $3^m.4$ to $4^m.5$, period $12^d\ 21^h\ 47^m$. Four distant companions.
18 49 .8	+32 54	*Ring Nebula in Lyra*, annular, gaseous, most interesting of its kind.
19 26 .7	+27 45	*β Cygni*, fine double, 3^m yellow and 7^m blue, $p=56°$, $s=35''$.

α, 1900	δ, 1900	Object: description: remarks.
$19^h\ 48^m.5$	$+70°\ \ 1'$	*ε Draconis*, double, $5^m.5$ and $9^m.5$, $s = 2''.8$.
19 55 .2	+22 27	*Nebula in Vulpecula*, the "Dumb Bell Nebula," double, large, probably gaseous.
20 14 .1	+37 43	*P Cygni*, variable, 3^m to 6^m.
20 42 .0	+15 46	*γ Delphini*, double, 4^m and 7^m, $s = 11''$.
20 58 .7	−11 45	*Nebula in Aquarius*, planetary, bright, very interesting in a large telescope.
21 2 .4	+38 15	61 *Cygni*, double, $5^m.5$ and 6^m, $s = 20''$, one of the nearest stars to us
21 8 .2	+68 5	*T Cephei*, variable, $5^m.6$ to $9^m.9$, period 383^d.
21 28 .2	− 1 16	*Cluster in Aquarius*, arge, globular.
22 23 .7	− 0 32	*ζ Aquarii*, double, 4^m and $4^m.5$, $s = 3''$.
23 21 .1	+41 59	*Nebula in Andromeda*, planetary, small, very bright, round.

TABLE I. PULCOVA REFRACTION TABLES.

App't z	log μ	λ	App't z	log μ	λ	App't z	log μ	λ	A
° ′			° ′			° ′			
0 0	1.76032		71 0	1.75614	1.0115	77 0	1.75131	1.0253	1.0029
5 0	1.76032		10	1.75606	1.0118	10	1.75107	1.0259	1.0029
10 0	1.76030		20	1.75598	1.0120	20	1.75083	1.0264	1.0030
15 0	1.76028		30	1.75590	1.0123	30	1.75058	1.0271	1.0030
20 0	1.76025		40	1.75582	1.0125	40	1.75032	1.0278	1.0031
25 0	1.76021		50	1.75573	1.0128	50	1.75005	1.0285	1.0032
30 0	1.76015		72 0	1.75564	1.0130	78 0	1.74976	1.0293	1.0033
35 0	1.76006		10	1.75555	1.0133	10	1.74947	1.0300	1.0033
40 0	1.75995		20	1.75546	1.0136	20	1.74917	1.0309	1.0034
45 0	1.75980	1.0018	30	1.75536	1.0138	30	1.74886	1.0318	1.0035
50 0	1.75960	1.0022	40	1.75526	1.0141	40	1.74853	1.0327	1.0036
51 0	1.75955	1.0024	50	1.75516	1.0144	50	1.74819	1.0335	1.0037
52 0	1.75949	1.0025	73 0	1.75506	1.0147	79 0	1.74783	1.0344	1.0038
53 0	1.75943	1.0026	10	1.75496	1.0150	10	1.74746	1.0354	1.0039
54 0	1.75936	1.0027	20	1.75485	1.0153	20	1.74707	1.0364	1.0040
55 0	1.75928	1.0029	30	1.75474	1.0157	30	1.74665	1.0374	1.0041
56 0	1.75920	1.0032	40	1.75462	1.0160	40	1.74623	1.0385	1.0042
57 0	1.75912	1.0035	50	1.75450	1.0163	50	1.74579	1.0397	1.0043
58 0	1.75902	1.0038	74 0	1.75438	1.0166	80 0	1.74533	1.0409	1.0044
59 0	1.75892	1.0041	10	1.75425	1.0170	10	1.74484	1.0421	1.0045
60 0	1.75881	1.0044	20	1.75412	1.0173	20	1.74433	1.0433	1.0046
61 0	1.75868	1.0047	30	1.75398	1.0177	30	1.74380	1.0447	1.0048
62 0	1.75853	1.0051	40	1.75384	1.0181	40	1.74325	1.0461	1.0049
63 0	1.75837	1.0055	50	1.75369	1.0185	50	1.74266	1.0475	1.0050
64 0	1.75820	1.0059	75 0	1.75354	1.0188	81 0	1.74204	1.0491	1.0052
65 0	1.75801	1.0064	10	1.75338	1.0191	10	1.74139	1.0508	1.0053
66 0	1.75780	1.0070	20	1.75322	1.0195	20	1.74071	1.0525	1.0055
67 0	1.75755	1.0077	30	1.75306	1.0200	30	1.73999	1.0542	1.0057
68 0	1.75727	1.0085	40	1.75289	1.0205	40	1.73924	1.0561	1.0059
69 0	1.75694	1.0093	50	1.75271	1.0211	50	1.73844	1.0580	1.0061
70 0	1.75657	1.0103	76 0	1.75253	1.0216	82 0	1.73760	1.0600	1.0063
10	1.75650	1.0105	10	1.75235	1.0223	10	1.73671	1.0622	1.0065
20	1.75643	1.0107	20	1.75216	1.0229	20	1.73577	1.0645	1.0068
30	1.75636	1.0109	30	1.75196	1.0235	30	1.73478	1.0669	1.0070
40	1.75629	1.0111	40	1.75175	1.0241	40	1.73373	1.0694	1.0073
50	1.75622	1.0113	50	1.75153	1.0246	50	1.73260	1.0720	1.0076
71 0	1.75614	1.0115	77 0	1.75131	1.0253	83 0	1.73143	1.0747	1.0078

SUPPLEMENT.

App't z	log μ tan z	λ	A	App't z	log μ tan z	λ	A
° ′				° ′			
82 30	2.61534	1.0669	1.0070	86 30	2.88535	1.1934	1.0203
83 0	2.64226	1.0747	1.0078	87 0	2.93113	1.2277	1.0241
83 30	2.67076	1.0839	1.0087	87 30	2.98087	1.2708	1.0294
84 0	2.70088	1.0949	1.0098	88 0	3.03519	1.3241	1.0357
84 30	2.73294	1.1080	1.0112	88 30	3.09458	1.3902	1.0437
85 0	2.76717	1.1235	1.0127	89 0	3.15994	1.4729	1.0541
85 30	2.80376	1.1424	1.0148	89 30	3.23206	1.5762	1.0680
86 0	2.84304	1.1652	1.0172	90 0	3.31186	1.7046	1.0859

TABLE I. PULCOVA REFRACTION TABLES.

B. Factor depending on the Barometer.

English inches.	log B	French metres.	log B	French metres.	log B.
27.5	—0.03191	0.725	—0.01560	0.760	+0.00488
27.6	—0.03033	0.726	—0.01500	0.761	0.00545
27.7	—0.02876	0.727	—0.01440	0.762	0.00602
27.8	—0.02720	0.728	—0.01380	0.763	0.00659
27.9	—0.02564	0.729	—0.01321	0.764	0.00716
28.0	—0.02409	0.730	—0.01261	0.765	0.00773
28.1	—0.02254	0.731	—0.01202	0.766	0.00830
28.2	—0.02099	0.732	—0.01142	0.767	0.00886
28.3	—0.01946	0.733	—0.01083	0.768	0.00943
28.4	—0.01793	0.734	—0.01024	0.769	0.00999
28.5	—0.01640	0.735	—0.00965	0.770	0.01056
28.6	—0.01488	0.736	—0.00906	0.771	0.01112
28.7	—0.01336	0.737	—0.00847	0.772	0.01168
28.8	—0.01185	0.738	—0.00788	0.773	0.01225
28.9	—0.01035	0.739	—0.00729	0.774	0.01281
29.0	—0.00885	0.740	—0.00670	0.775	0.01337
29.1	—0.00735	0.741	—0.00612	0.776	0.01393
29.2	—0.00586	0.742	—0.00553	0.777	0.01449
29.3	—0.00438	0.743	—0.00494	0.778	0.01505
29.4	—0.00290	0.744	—0.00436	0.779	0.01560
29.5	—0.00142	0.745	—0.00378	0.780	0.01616
29.6	+0.00005	0.746	—0.00319	0.781	0.01672
29.7	0.00151	0.747	—0.00261	0.782	0.01727
29.8	0.00297	0.748	—0.00203	0.783	0.01783
29.9	0.00443	0.749	—0.00145	0.784	0.01838
30.0	0.00588	0.750	—0.00087	0.785	0.01894
30.1	0.00732	0.751	—0.00029	0.786	0.01949
30.2	0.00876	0.752	+0.00028	0.787	0.02004
30.3	0.01020	0.753	0.00086	0.788	0.02059
30.4	0.01163	0.754	0.00144	0.789	0.02114
30.5	0.01306	0.755	0.00201	0.790	0.02169
30.6	0.01448	0.756	0.00259	0.791	0.02224
30.7	0.01589	0.757	0.00316	0.792	0.02279
30.8	0.01731	0.758	0.00374	0.793	0.02334
30.9	0.01871	0.759	0.00431	0.794	0.02389
31.0	+0.02012	0.760	+0.00488	0.795	+0.02443

T. Factor depending on Attached Thermometer.

Fahr.	log T.	Cent.	log T.
—20°	+0.00201	—30°	+0.00209
—10	0.00162	—25	0.00174
0	0.00123	—20	0.00139
+10	0.00085	—15	0.00104
20	0.00047	—10	0.00069
30	+0.00008	— 5	+0.00035
40	—0.00030	0	0.00000
50	—0.00069	+5	—0.00035
60	—0.00108	10	—0.00069
70	—0.00146	15	—0.00104
80	—0.00184	20	—0.00138
90	—0.00222	25	—0.00173
100	—0.00262	+30	—0.00207

TABLE I. PULCOVA REFRACTION TABLES.

γ. *Factor depending on External Thermometer.*

Fahr.	log γ	Fahr.	log γ	Cent.	log γ
−22°	+0.06560	+35°	+0.01200	−30°	+0.06560
−21	0.06461	36	0.01112	−29	0.06381
−20	0.06361	37	0.01023	−28	0.06202
−19	0.06262	38	0.00935	−27	0.06023
−18	0.06162	39	0.00848	−26	0.05846
−17	0.06063	40	0.00760	−25	0.05669
−16	0.05964	41	0.00672	−24	0.05493
−15	0.05866	42	0.00585	−23	0.05317
−14	0.05767	43	0.00498	−22	0.05142
−13	0.05669	44	0.00411	−21	0.04968
−12	0.05571	45	0.00324	−20	0.04795
−11	0.05473	46	0.00238	−19	0.04622
−10	0.05376	47	0.00151	−18	0.04451
− 9	0.05279	48	+0.00064	−17	0.04279
− 8	0.05182	49	−0.00022	−16	0.04108
− 7	0.05085	50	−0.00107	−15	0.03938
− 6	0.04988	51	−0.00193	−14	0.03769
− 5	0.04891	52	−0.00279	−13	0.03601
− 4	0.04795	53	−0.00364	−12	0.03433
− 3	0.04699	54	−0.00449	−11	0.03265
− 2	0.04603	55	−0.00535	−10	0.03099
− 1	0.04508	56	−0.00620	− 9	0.02933
0	0.04413	57	−0.00704	− 8	0.02767
+ 1	0.04318	58	−0.00789	− 7	0.02602
2	0.04223	59	−0.00873	− 6	0.02438
3	0.04128	60	−0.00957	− 5	0.02274
4	0.04033	61	−0.01041	− 4	0.02112
5	0.03938	62	−0.01125	− 3	0.01950
6	0.03844	63	−0.01209	− 2	0.01788
7	0.03750	64	−0.01293	− 1	0.01627
8	0.03657	65	−0.01376	0	0.01466
9	0.03563	66	−0.01459	+ 1	0.01306
10	0.03470	67	−0.01543	2	0.01147
11	0.03377	68	−0.01626	3	0.00988
12	0.03284	69	−0.01709	4	0.00830
13	0.03191	70	−0.01792	5	0.00672
14	0.03099	71	−0.01874	6	0.00515
15	0.03007	72	−0.01956	7	0.00359
16	0.02915	73	−0.01838	8	0.00203
17	0.02822	74	−0.01920	9	+0.00047
18	0.02730	75	−0.02202	10	−0.00107
19	0.02639	76	−0.02284	11	−0.00261
20	0.02548	77	−0.02366	12	−0.00415
21	0.02456	78	−0.02447	13	−0.00569
22	0.02364	79	−0.02528	14	−0.00721
23	0.02273	80	−0.02609	15	−0.00873
24	0.02183	81	−0.02690	16	−0.01025
25	0.02094	82	−0.02771	17	−0.01176
26	0.02004	83	−0.02851	18	−0.01326
27	0.01914	84	−0.02932	19	−0.01476
28	0.01824	85	−0.03012	20	−0.01626
29	0.01734	86	−0.03093	21	−0.01775
30	0.01645	87	−0.03173	22	−0.01923
31	0.01555	88	−0.03253	23	−0.02071
32	0.01466	89	−0.03333	24	−0.02219
33	0.01377	90	−0.03413	25	−0.02366
34	0.01288	91	−0.03492	30	−0.03093
+35	0.01200	+92	−0.03572	+35	−0.03810

TABLE I. PULCOVA REFRACTION TABLES.

log σ.

App't z	log σ	App't z	log σ
° ′		° ′	
80 0	0.00019	85 0	0.00146
80 30	0.00022	85 30	0.00185
81 0	0.00025	86 0	0.00241
81 30	0.00029	86 30	0.00320
82 0	0.00035	87 0	0.00421
82 30	0.00045	87 30	0.00561
83 0	0.00057	88 0	0.00749
83 30	0.00073	88 30	0.01006
84 0	0.00091	89 0	0.01352
84 30	0.00116	89 30	0.01813
85 0	0.00146	90 0	0.02424

i.

Date.	i
Jan. 15	+0.34
Feb. 15	+0.27
Mar. 15	+0.05
April 15	—0.08
May 15	—0.20
June 15	—0.26
July 15	—0.33
Aug. 15	—0.30
Sept. 15	—0.19
Oct 15	+0.16
Nov. 15	+0.33
Dec. 15	+0.37

$$\log r = \log \mu + \log \tan z + A\,(\log B + \log T) + \lambda \log \gamma + i \log \sigma$$

TABLE II. PULCOVA MEAN REFRACTIONS.

Barom 29.5 inches, Att. Therm. 50° F., Ext. Therm. 50° F.

App't z	Mean refr'n.	App't z	Mean refr'n.	App't z	Mean refr'n.	App't z	Mean refr'n.
° ′	′ ″	° ′	′ ″	° ′	′ ″	° ′	′ ″
0 0	0 0.0	58 0	1 31.2	73 0	3 4.7	80 40	5 34
5 0	0 5.0	59 0	1 34.8	73 20	3 8.5	81 0	5 46
10 0	0 10.1	60 0	1 38.7	73 40	3 12.5	81 20	5 58
15 0	0 15.3	61 0	1 42.8	74 0	3 16.6	81 40	6 12
20 0	0 20.8	62 0	1 47.1	74 20	3 20.9	82 0	6 26
25 0	0 26.7	63 0	1 51.7	74 40	3 25.4	82 20	6 41
30 0	0 33.0	64 0	1 56.6	75 0	3 30.0	82 40	6 58
32 0	0 35.7	65 0	2 1.9	75 20	3 34.8	83 0	7 15
34 0	0 38.5	65 30	2 4.7	75 40	3 39.9	83 20	7 35
36 0	0 41.5	66 0	2 7.6	76 0	3 45.2	83 40	7 56
38 0	0 44.6	66 30	2 10.6	76 20	3 50.7	84 0	8 19
40 0	0 47.9	67 0	2 13.8	76 40	3 56.5	84 20	8 43
42 0	0 51.4	67 30	2 17.1	77 0	4 2.5	84 40	9 10
44 0	0 55.1	68 0	2 20.5	77 20	4 8.8	85 0	9 40
46 0	0 59.1	68 30	2 24.1	77 40	4 15.5	85 30	10 32
48 0	1 3.4	69 0	2 27.8	78 0	4 22.5	86 00	11 31
50 0	1 8.0	69 30	2 31.7	78 20	4 29.8	86 30	12 42
52 0	1 13.0	70 0	2 35.7	78 40	4 37.6	87 0	14 7
53 0	1 15.7	70 30	2 39.9	79 0	4 45.7	87 30	15 49
54 0	1 18.5	71 0	2 44.4	79 20	4 54.4	88 0	17 55
55 0	1 21.4	71 30	2 49.1	79 40	5 3.5	88 30	20 33
56 0	1 24.5	72 0	2 54.0	80 0	5 13.1	89 0	23 53
57 0	1 27.8	72 30	2 59.2	80 20	5 23.4	89 30	28 11
58 0	1 31.2	73 0	3 4.7	80 40	5 34.3	90 0	33 51

TABLE III. $m = \dfrac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$, or $m = \dfrac{2 \sin^2 \frac{1}{2} (t - t_0)}{\sin 1''}$.

t, or $t - t_0$	m	t, or $t - t_0$	m	t, or $t - t_0$	m	t, or $t - t_0$	m	t, or $t - t_0$	m
m s	″	m s	″	m s	″	m s	″	m s	″
0 0	0.00	4 0	31.42	8 0	125.65	12 0	282.68	16 0	502.5
0 5	0.01	4 5	32.74	8 5	128.28	12 5	286.62	16 5	507.7
0 10	0.05	4 10	34.09	8 10	130.94	12 10	290.58	16 10	513.0
0 15	0.12	4 15	35.46	8 15	133.63	12 15	294.58	16 15	518.3
0 20	0.22	4 20	36.87	8 20	136.34	12 20	298.60	16 20	523.6
0 25	0.34	4 25	38.30	8 25	139.08	12 25	302.64	16 25	529.0
0 30	0.49	4 30	39.76	8 30	141.85	12 30	306.72	16 30	534.3
0 35	0.67	4 35	41.25	8 35	144.64	12 35	310.82	16 35	539.7
0 40	0.87	4 40	42.76	8 40	147.46	12 40	314.95	16 40	545.2
0 45	1.10	4 45	44.30	8 45	150.31	12 45	319.10	16 45	550.6
0 50	1.36	4 50	45.87	8 50	153.19	12 50	323.29	16 50	556.1
0 55	1.65	4 55	47.46	8 55	156.09	12 55	327.50	16 55	561.6
1 0	1.96	5 0	49.09	9 0	159.02	13 0	331.74	17 0	567.2
1 5	2.31	5 5	50.73	9 5	161.98	13 5	336.00	17 5	572.8
1 10	2.67	5 10	52.41	9 10	164.97	13 10	340.30	17 10	578.4
1 15	3.07	5 15	54.11	9 15	167.97	13 15	344.62	17 15	584.0
1 20	3.49	5 20	55.84	9 20	171.02	13 20	348.97	17 20	589.6
1 25	3.94	5 25	57.60	9 25	174.08	13 25	353.34	17 25	595.3
1 30	4.42	5 30	59.40	9 30	177.18	13 30	357.74	17 30	601.0
1 35	4.92	5 35	61.20	9 35	180.30	13 35	362.17	17 35	606.8
1 40	5.45	5 40	63.05	9 40	183.46	13 40	366.64	17 40	612.5
1 45	6.01	5 45	64.91	9 45	186.63	13 45	371.11	17 45	618.3
1 50	6.60	5 50	66.81	9 50	189.83	13 50	375.12	17 50	624.1
1 55	7.21	5 55	68.73	9 55	193.06	13 55	380.17	17 55	630.0
2 0	7.85	6 0	70.68	10 0	196.32	14 0	384.74	18 0	635.9
2 5	8.52	6 5	72.66	10 5	199.60	14 5	389.32	18 5	641.7
2 10	9.22	6 10	74.66	10 10	202.92	14 10	393.94	18 10	647.7
2 15	9.94	6 15	76.69	10 15	206.26	14 15	398.58	18 15	653.6
2 20	10.69	6 20	78.75	10 20	209.62	14 20	403.26	18 20	659.6
2 25	11.47	6 25	80.84	10 25	213.02	14 25	407.96	18 25	665.6
2 30	12.27	6 30	82.95	10 30	216.44	14 30	412.68	18 30	671.6
2 35	13.10	6 35	85.09	10 35	219.88	14 35	417.44	18 35	677.7
2 40	13.96	6 40	87.26	10 40	223.36	14 40	422.23	18 40	683.8
2 45	14.85	6 45	89.45	10 45	226.86	14 45	427.04	18 45	689.9
2 50	15.76	6 50	91.68	10 50	230.39	14 50	431.87	18 50	696.0
2 55	16.70	6 55	93.92	10 55	233.95	14 55	436.73	18 55	702.2
3 0	17.67	7 0	96.20	11 0	237.54	15 0	441.63	19 0	708.4
3 5	18.67	7 5	98.50	11 5	241.14	15 5	446.55	19 5	714.6
3 10	19.69	7 10	100.84	11 10	244.79	15 10	451.50	19 10	720.9
3 15	20.74	7 15	103.20	11 15	248.45	15 15	456.47	19 15	727.2
3 20	21.82	7 20	105.58	11 20	252.15	15 20	461.47	19 20	733.5
3 25	22.92	7 25	107.99	11 25	255.87	15 25	466.50	19 25	739.8
3 30	24.05	7 30	110.44	11 30	259.62	15 30	471.55	19 30	746.2
3 35	25.21	7 35	112.90	11 35	263.39	15 35	476.64	19 35	752.6
3 40	26.40	7 40	115.40	11 40	267.20	15 40	481.74	19 40	759.0
3 45	27.61	7 45	117.92	11 45	271.02	15 45	486.88	19 45	765.4
3 50	28.85	7 50	120.47	11 50	274.88	15 50	492.05	19 50	771.9
3 55	30.12	7 55	123.05	11 55	278.76	15 55	497.23	19 55	778.4
4 0	31.42	8 0	125.65	12 0	282.68	16 0	502.46	20 0	784.9

TABLE III. $m = \dfrac{2\sin^2 \frac{1}{2} t}{\sin 1''}$, or $m = \dfrac{2\sin^2 \frac{1}{2}(t - t_0)}{\sin 1''}$.

t, or $t-t_0$		m	t, or $t-t_0$		m
m	*s*	″	*m*	*s*	″
20	0	784.9	24	0	1129.9
20	5	791.4	24	5	1137.8
20	10	798.0	24	10	1145.6
20	15	804.6	24	15	1153.6
20	20	811.3	24	20	1161.5
20	25	817.9	24	25	1169.5
20	30	824.6	24	30	1177.5
20	35	831.2	24	35	1185.5
20	40	838.0	24	40	1193.5
20	45	844.7	24	45	1201.5
20	50	851.6	24	50	1209.6
20	55	858.4	24	55	1217.7
21	0	865.3	25	0	1225.9
21	5	872.1	25	5	1234.1
21	10	879.0	25	10	1242.3
21	15	886.0	25	15	1250.5
21	20	893.0	25	20	1258.8
21	25	900.0	25	25	1267.1
21	30	907.0	25	30	1275.4
21	35	914.0	25	35	1283.8
21	40	921.1	25	40	1292.2
21	45	928.2	25	45	1300.5
21	50	935.2	25	50	1309.0
21	55	942.3	35	55	1317.4
22	0	949.5	26	0	1325.9
22	5	956.7	26	5	1334.4
22	10	963.9	26	10	1342.9
22	15	971.2	26	15	1351.4
22	20	978.5	26	20	1360.1
22	25	985.8	26	25	1368.7
22	30	993.2	26	30	1377.3
22	35	1000.6	26	35	1385.9
22	40	1008.0	26	40	1394.7
22	45	1015.4	26	45	1403.4
22	50	1022.8	26	50	1412.2
22	55	1030.3	26	55	1420.9
23	0	1037.8	27	0	1429.7
23	5	1045.3	27	5	1438.5
23	10	1052.8	27	10	1447.4
23	15	1060.4	27	15	1456.3
23	20	1068.1	27	20	1465.2
23	25	1075.7	27	25	1474.1
23	30	1083.3	27	30	1483.1
23	35	1091.0	27	35	1492.1
23	40	1098.8	27	40	1501.1
23	45	1106.5	27	45	1510.2
23	50	1114.3	27	50	1519.2
23	55	1122.0	27	55	1528.3
24	0	1129.9	28	0	1537.5

$n = \dfrac{2\sin^4 \frac{1}{2} t}{\sin 1''}$.

t		n
m	*s*	″
0	0	0.00
2	0	0.00
4	0	0.00
6	0	0.01
8	0	0.04
9	0	0.06
10	0	0.09
11	0	0.14
12	0	0.19
12	30	0.23
13	0	0.26
13	30	0.31
14	0	0.36
14	30	0.41
15	0	0.47
15	30	0.54
16	0	0.61
16	30	0.69
17	0	0.78
17	30	0.88
18	0	0.98
18	30	1.09
19	0	1.22
19	30	1.35
20	0	1.49
20	20	1.60
20	40	1.70
21	0	1.82
21	20	1.93
21	40	2.06
22	0	2.19
22	20	2.32
22	40	2.46
23	0	2.61
23	20	2.77
23	40	2.93
24	0	3.10
24	20	3.27
24	40	3.45
25	0	3.64
25	20	3.84
25	40	4.05
26	0	4.26
26	20	4.48
26	40	4.72
27	0	4.96
27	20	5.20
27	40	5.46
28	0	5.73